VOLUME 13 (1959) No. 5

# NUCLEAR PHYSICS

JOURNAL DEVOTED TO THE EXPERIMENTAL AND THEORETICAL STUDY OF THE FUNDAMENTAL CONSTITUENTS OF MATTER AND THEIR INTERACTIONS

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**NUCLEAR PHYSICS** is published in fortnightly issues. The subscription price per volume of 700 pages is fl 57.00; 114s; $ 15.00. Post-free.

**SUBSCRIPTIONS** should be sent to the publishers, **NORTH-HOLLAND PUBLISHING COMPANY**, P.O.Box 103, Amsterdam or to any subscription-agent.

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**Printed in the Netherlands**

1.C:2.E *Nuclear Physics* **13** (1959) 621—641; © *North-Holland Publishing Co., Amsterdam*
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# THE MEDIUM AND HIGH ENERGY OPTICAL MODEL AND THE CORRELATIONS IN THE NUCLEAR WAVE FUNCTION

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Received 10 June 1959

**Abstract:** Watson's multiple scattering theory is applied to calculate the well depths of the nuclear optical potential for the case of strongly correlated nuclear matter. The nuclear ground state is represented by a Jastrow type wave function. The consequences of the Bethe-Goldstone equation for the correlation functions are considered. Our numerical results obtained for the case of a separable nucleon-nucleon interaction show that the effects of the correlations amount to corrections to the Fermi model case up to the order of 10 per cent in the medium and high energy range. The evaluated corrections due to the incoherent multiple scattering appear to be important.

## 1. Introduction

The recent developments of the theory of the nuclear many-body problem render it possible to have some more qualitative and even quantitative insight into the nature of the nucleon-nucleus scattering. It now seems possible to attempt some explicit calculations along the lines of Watson's theory of the optical model potential [1–4]). Our present better knowledge of the nucleon-nucleon scattering and of the gross properties of nuclei leads to a better insight into the effective nucleon-nucleon scattering operators: the $t$-matrices and the multiple scattering operators. On the other hand, our knowledge of the ground-state nuclear wave function has been increased and we can provide some quantitatively reasonable approximations for the effect of the strong correlations within nuclear matter.

In the asymptotic region of high energies of the incoming nucleon some of the effects of the identity of the projectile and the target particles [3]) and the effects of the self-consistency become of little importance and the impulse approximation of Riesenfeld and Watson [5]) becomes justified.

In the medium, and especially in the low, energy range a number of difficulties make this problem very involved. The effects of the exclusion

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††† Supported in part by the U. S. Atomic Energy Commission and the Higgins Scientific Trust Fund.



November (3) 1959

principle on the $t$-matrix, together with the self-consistency effects involving the rearrangement energy problem, render the expression for the effective nucleon-nucleus potential hardly tractable except in the very low energy limit where the Brueckner approach has been used [6,7]). This approach, however, in which the $t$-matrices are replaced by the $K$-matrices and the centre of mass momentum of the two colliding nucleons is neglected, may only be justified in the very low energy limit.

If one confines oneself, however, to the medium and high energies † it seems reasonable not to treat the target nucleons and the projectile on the same footing. One can first compute the $t$-matrix assuming the distinguishability of the projectile and then anti-symmetrize it a posteriori with respect to the two particles involved (the projectile and the target nucleon considered), as was indicated in ref. [3]). One may go further and take into account the presence of the target nucleons other than the one involved in an instantaneous scattering with the projectile. This is in fact done in the present paper by excluding from the integral equation for the not yet anti-symmetrized $t$-matrix all the intermediate states occupied by the other target nucleons. In this procedure the centre of mass motion of the two colliding nucleons is taken into account by integration in the intermediate states outside a complicated domain in the momentum space, as the $t$-matrix depends essentially on the absolute momentum of the pair of particles involved.

Recently, Verlet and Gavoret [8]) presented this type of calculation using the separable nucleon-nucleon potentials [9]) and averaging the computed $t$-matrices over the uncorrelated wave functions of the Fermi model. The use of the separable nucleon-nucleon interaction has the advantage of permitting an exact calculation of the $t$-matrix. This enables one to draw some general conclusions about the optical model even if the separable interaction is not very realistic.

In the present paper we shall use the same potentials in our numerical computations to simplify the calculations and to have a direct comparison with the results of Verlet and Gavoret.

Watson's theory [2]) provides the following expression for the nucleon-nucleus potential:

$$V = \sum_{i=1}^{A} \langle \psi_0 | t_{0i} G_{0i} | \psi_0 \rangle, \tag{1}$$

where $\psi_0$ is the ground state wave function of the target nucleus and where the effective interaction between the projectile 0 and the target nucleon $i$

† It is difficult to define precisely the low energy limit of the applicability of the present approach. However, the analysis by Takeda and Watson[3]) shows that for energies sufficiently high (more than 50—100 MeV) for the impulse approximation to be valid, the approach followed in this paper should also be valid.

has to be antisymmetrized with respect to the particles 0 and $i$. The scattering operator $t_{0i}$ describing the scattering between the particles 0 and $i$ in the presence of the other target nucleons satisfies the integral equation:

$$t_{0i} = V_{0i} + V_{0i} \frac{Q}{e} t_{0i}, \tag{2}$$

where the operator $Q$ excludes all the states occupied by the target nucleons, and

$$e = E - H_A - T_0 - A t_{c0i} + i\eta \tag{3}$$

where $V_{0i}$ is the nucleon-nucleon interaction, $E$ is the ground state energy of the target nucleus plus the energy of the projectile, $H_A$ is the hamiltonian of the target nucleus, $T_0$ the kinetic energy of the projectile and $t_{c0i}$ is the coherent part of the $t_{0i}$ operator, i.e. the part diagonal in the states of the target nucleus: $t_{0i} = t_{c0i} + I_{0i}$. The incoherent scattering operator

$$G_{0i} - 1 = \frac{Q}{e} \sum_{j \neq i}^{A} I_{0j} G_{0j} \tag{4}$$

describes the effect of the inelastic scattering by the other target particles. Throughout our calculations we put $G_{0i} - 1 \approx 0$; however, an estimate of the incoherent scattering corrections is given subsequently.

The main task of the present paper is to discuss the effects of the correlations in the ground state nuclear wave function $\psi_0$ on the optical potential $V$. This is especially interesting at higher energies, since it is well known [10]) that the correlations are quite essential for the description of many high energy nuclear reactions. The particular importance of these effects for the pick-up reaction was pointed out by Heidmann [11]) and for the nuclear photo-effect by Levinger [12]).

In the present paper we confine ourselves to the infinite nuclear matter and consequently we consider the correlations introduced in the Fermi gas model. The Fermi gas model was modified by Jastrow [13]) who introduced product correlation functions taking account of the repulsive core in the two-body interaction. Many authors have developed this method and used various forms of the correlation functions as variational trial functions in computing the binding energy per particle. Furthermore one can show [14]) that if one tries to satisfy the many-body Schrödinger equation with the Jastrow wave function one gets equations of the Bethe-Goldstone[15]) type for the correlation functions. In this procedure it turns out that it is necessary to generalize the Jastrow correlation functions by making them state dependent.

Recently de Shalit and Weisskopf [16]) presented a discussion of the wave function $\psi_0$ involving state dependent correlations of the form

$$\psi_0 = \frac{1}{\sqrt{N\,A!}} \sum_P \varepsilon_P (1+\lambda_{p_1 p_2}(12)) \ldots (1+\lambda_{p_{A-1} p_A}(A-1, A)) \times \phi_{p_1}(1)\phi_{p_2}(2) \ldots \phi_{p_A}(A) \tag{5}$$

where $\phi_{p_i}(i)$ is a single particle orbital (a plane wave in the Fermi model), $N$ is the normalization constant and the sum is extended over all the permutations $P = \binom{1 \ldots i \ldots A}{p_1 \ldots p_i \ldots p_A}$ with the proper signs $\varepsilon_P$. It can be shown that $\psi_0$ of eq. (5) may be a good approximation to the actual nuclear wave function provided the $(1+\lambda)$'s correspond to the solutions of the appropriate Bethe-Goldstone equations †. In the following we shall mainly concentrate on this form of the wave function $\psi_0$.

In section 2 we shall present the general calculation and a discussion of the cluster development resulting from the use of the correlated $\psi_0$. The numerical results and their discussion are given in section 3. Finally in section 4 we give a rough estimate of the incoherent scattering corrections.

## 2. Calculation

The starting point for the calculation of the optical potential $V$ is eq. (1) given by Watson's theory. Throughout this calculation we neglect the incoherent scattering and put

$$G_{0i} \approx 1. \tag{6}$$

This approximation will be discussed in section 4.

To calculate $V$ in momentum representation, i.e.

$$\langle \mathbf{k}'_0|V|\mathbf{k}_0\rangle = \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{r}_0 \int d\sigma_0 \int d\tau_0 \eta^*_{s_0}(\sigma_0)\chi^*_{t_0}(\tau_0) e^{-i\mathbf{k}'_0 \cdot \mathbf{r}_0} \times V \eta_{s_0}(\sigma_0)\chi_{t_0}(\tau_0) e^{i\mathbf{k}_0 \cdot \mathbf{r}_0} \tag{7}$$

($\eta_s$ and $\chi_t$ are the spin and isospin functions; $s$ and $t$ are the corresponding "magnetic numbers"), we write $\psi_0$ in the form

$$\psi_0(1, 2 \ldots A) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_{s_1 t_1} \int d\mathbf{k}_1 e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} \eta_{s_1}(\sigma_1)\chi_{t_1}(\tau_1)\bar{\psi}_0(\mathbf{k}_1 s_1 t_1, 2 \ldots A). \tag{8}$$

After substituting this form of $\psi_0$ into eq. (7) with $V$ defined by eq. (1) a straightforward calculation gives the following result for the generally non-local optical potential in the momentum representation:

$$\langle \mathbf{k}'_0|V|\mathbf{k}_0\rangle = \int d\mathbf{k}_1 d\mathbf{k}'_1 \langle \mathbf{k}'_{01}|\bar{t}|\mathbf{k}_{01}\rangle \delta(\mathbf{K}_{01} - \mathbf{K}'_{01}) \times \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{r}_a d\mathbf{r}_b e^{i(\mathbf{k}'_1 \cdot \mathbf{r}_a - \mathbf{k}_1 \cdot \mathbf{r}_b)} \rho(\mathbf{r}_a, \mathbf{r}_b) \tag{9}$$

† However, quite recently Lüders [17]) put forward some objections against the proof given in ref.[17]). Some of the difficulties connected with the BG equation were first pointed out by Gottfried [18]).

where the one particle mixed density of the ground state of the target nucleus

$$\rho(\mathbf{r}_a, \mathbf{r}_b) \equiv A\int \mathrm{d}\sigma_1 \mathrm{d}\tau_1 \mathrm{d}(2) \ldots \mathrm{d}(i) \ldots \mathrm{d}(A) \psi_0^*(\mathbf{r}_a \sigma_1 \tau_1, 2 \ldots i \ldots A) \times \psi_0(\mathbf{r}_b \sigma_1 \tau_1, 2 \ldots i \ldots A). \tag{10}$$

In these formulae, $i$ stands for $\mathbf{r}_i$, $\sigma_i$, $\tau_i$, the space, spin and isospin coordinates lumped together into one symbol; $\mathbf{k}_0$, $\mathbf{k'}_0$ and $\mathbf{k}_1$, $\mathbf{k'}_1$ are the initial and final momenta of the projectile and of the target particle, respectively, and

$$\mathbf{k}_{01} = \tfrac{1}{2}(\mathbf{k}_0 - \mathbf{k}_1), \quad \mathbf{k'}_{01} = \tfrac{1}{2}(\mathbf{k'}_0 - \mathbf{k'}_1), \quad \mathbf{K}_{01} = \mathbf{k}_0 + \mathbf{k}_1, \quad \mathbf{K'}_{01} = \mathbf{k'}_0 + \mathbf{k'}_1. \tag{11}$$

Finally $\langle \mathbf{k'}_{01} | \bar{t} | \mathbf{k}_{01} \rangle$ is the "effective" antisymmetrized and self-consistent average $t$-matrix †. Writing the singlet (s) and triplet (tr) parts of $\bar{t}$ explicitly and assuming the charge independence of nuclear forces we have ††

$$\langle \mathbf{k}' | \bar{t} | \mathbf{k} \rangle = \tfrac{3}{4}\langle \mathbf{k}' | t | \mathbf{k} \rangle_{(\mathrm{tr})} + \tfrac{1}{4}\langle \mathbf{k}' | t | \mathbf{k} \rangle_{(\mathrm{s})} - \tfrac{3}{8}\langle \mathbf{k}' | t | -\mathbf{k} \rangle_{(\mathrm{tr})} + \tfrac{1}{8}\langle \mathbf{k}' | t | -\mathbf{k} \rangle_{(\mathrm{s})}. \tag{12}$$

The exclusion principle, i.e. the occurrence of $Q$ in eq. (2) and the self-consistency effects make the $\bar{t}$-matrix $\mathbf{k}(= \mathbf{k}_{01})$-dependent. This dependence on $\mathbf{K}$ is not indicated explicitly in our notation.

In the case of an infinite medium (nuclear matter) $\rho(\mathbf{r}_a, \mathbf{r}_b)$ is a function of the difference $|\mathbf{r}_a - \mathbf{r}_b| = r_{ab}$ only, $\rho(\mathbf{r}_a, \mathbf{r}_b) = \rho(r_{ab})$, and the optical potential, eq. (9), becomes diagonal in the momentum space:

$$\langle \mathbf{k'}_0 | V | \mathbf{k}_0 \rangle = \delta(\mathbf{k'}_0 - \mathbf{k}_0) V(k_0) \tag{13}$$

where

$$V(k_0) = \int \mathrm{d}\mathbf{k}_1 \bar{t}(\mathbf{k}_{01}) \int \mathrm{d}\mathbf{r}_{ab} \mathrm{e}^{-i\mathbf{k}_1 \cdot \mathbf{r}_{ab}} \rho(r_{ab}), \tag{14}$$

where $\bar{t}(\mathbf{k}_{01}) = \langle \mathbf{k}_{01} | \bar{t} | \mathbf{k}_{01} \rangle$ is the on-the-energy-shell forward scattering component of the effective $t$-matrix. As a result of $\mathbf{k'}_{01} = \mathbf{k}_{01}$ the potential given by eq. (14) does not involve any spin-orbit part which vanishes in this case as proportional to $\mathbf{k}_{01} \times \mathbf{k'}_{01}$.

Let us first consider the Jastrow [13]) type wave function

$$\psi_0 = \frac{1}{\sqrt{N!}} \prod_{i>j}^{A} [1 + \lambda(r_{ij})] \sum_P \varepsilon_P \prod_{l=1}^{A} \phi_{P_l}(l) \tag{15}$$

with

$$\phi_n(l) = \eta_{s_n}(\sigma_l) \chi_{t_n}(\tau_l) \varphi_{\alpha_n}(\mathbf{r}_l), \tag{16}$$

† The energy denominator $e$ of eq. (3) was replaced by a sum of the appropriate self-consistent single particle hamiltonians.

†† If $t$ operates differently in even and odd states we would have

a) direct terms: $\langle \mathbf{k}' | t | \mathbf{k} \rangle_{(\mathrm{tr})(\mathrm{s})} = \langle \mathbf{k}' | t_{\mathrm{e}} | \mathbf{k} \rangle_{(\mathrm{tr})(\mathrm{s})} + \langle \mathbf{k}' | t_{\mathrm{o}} | \mathbf{k} \rangle_{(\mathrm{tr})(\mathrm{s})}$,

b) exchange terms: $\langle \mathbf{k}' | t | -\mathbf{k} \rangle_{(\mathrm{tr})(\mathrm{s})} = \langle \mathbf{k}' | t_{\mathrm{e}} | \mathbf{k} \rangle_{(\mathrm{tr})(\mathrm{s})} - \langle \mathbf{k}' | t_{\mathrm{o}} | \mathbf{k} \rangle_{(\mathrm{tr})(\mathrm{s})}$,

where $t_{\mathrm{e}}$ and $t_{\mathrm{o}}$ contain the projection operators into the even and odd states, respectively.

where the state independent correlation function $\lambda(r_{ij})$ has the property $\lambda(r_{ij}) \to 0$ for $r_{ij} \to \infty$ and in the case of a repulsive core of the radius $r_c$ in the nucleon-nucleon interaction it satisfies the boundary condition $\lambda(r_{ij}) = -1$ for $r_{ij} \leqq r_c$. Variational trial wave functions of this form have been discussed, e.g. by Iwamoto and Yamada [19]), Emery [20]), Aviles [21]), Hartogh and Tolhoek [22]) for nuclear matter, and by Dąbrowski [23]) and Sawicki and Folk [24]) for a finite nucleus.

For the wave function (15) we can perform the Ursell-Mayer cluster development of $\rho(\mathbf{r}_a, \mathbf{r}_b)$ (compare Hartogh and Tolhoek [22])). We find

$$\rho(\mathbf{r}_a, \mathbf{r}_b) = \rho_0(\mathbf{r}_a, \mathbf{r}_b)+\rho_1(\mathbf{r}_a, \mathbf{r}_b)+\rho_2(\mathbf{r}_a, \mathbf{r}_b)+ \text{higher cluster terms} \quad (17)$$

where the uncorrelated mixed density

$$\rho_0(\mathbf{r}_a, \mathbf{r}_b) = 4 \sum_{\alpha} \varphi_\alpha^*(\mathbf{r}_a)\varphi_\alpha(\mathbf{r}_b), \quad (18)$$

and where

$$\rho_1(\mathbf{r}_a, \mathbf{r}_b) = \int d\mathbf{r}_2 h(r_{2a}, r_{2b})\{\rho_0(\mathbf{r}_a, \mathbf{r}_b)\rho_0(\mathbf{r}_2) - \tfrac{1}{4}\rho_0(\mathbf{r}_a, \mathbf{r}_2)\rho_0(\mathbf{r}_2, \mathbf{r}_b) \quad (19)$$

$$\rho_2(\mathbf{r}_a, \mathbf{r}_b) = -\tfrac{1}{4}\int d\mathbf{r}_2 \int d\mathbf{r}_3 h(r_{23})\{\rho_0(\mathbf{r}_a, \mathbf{r}_2)\rho_0(\mathbf{r}_2, \mathbf{r}_b)\rho_0(\mathbf{r}_3) - \tfrac{1}{4}\rho_0(\mathbf{r}_a, \mathbf{r}_2)\rho_0(\mathbf{r}_2, \mathbf{r}_3)\rho_0(\mathbf{r}_3, \mathbf{r}_b)\}. \quad (20)$$

The $h$ functions are defined by

$$\begin{aligned} h(r) &= |1+\lambda(r)|^2 - 1 = |\lambda(r)|^2 + \lambda(r) + \lambda^*(r), \\ h(r_{2a}, r_{2b}) &= [1+\lambda(r_{2a})]^*[1+\lambda(r_{2b})] - 1. \end{aligned} \quad (21)$$

The single particle density $\rho_0(\mathbf{r}_i) = \rho_0(\mathbf{r}_i, \mathbf{r}_i)$. We apply the vector notation $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$.

From now on we confine ourselves to the case of nuclear matter. In this case the functions $\varphi$'s are the normalized plane waves $\Omega^{-\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{r}}$ ($\Omega$ = volume of the system) with momenta ranging from zero up to the Fermi momentum $\hbar k_F$ and we have

$$\rho_0(\mathbf{r}_a, \mathbf{r}_b) = \frac{4}{(2\pi)^3}\int^{k_F} d\boldsymbol{\varkappa} e^{i\boldsymbol{\varkappa}\cdot(\mathbf{r}_b - \mathbf{r}_a)} = \rho_0 l(r_{ab}), \quad (22)$$

where

$$l(r_{ab}) = 3j_1(k_F r_{ab})/k_F r_{ab} \quad (23)$$

and

$$\rho_0 = \rho_0(\mathbf{r}_i) = \frac{A}{\Omega} = \frac{2k_F^3}{3\pi^2}. \quad (24)$$

To the cluster development of the density $\rho(\mathbf{r}_a, \mathbf{r}_b)$, eq. (17), there corresponds the following development of the optical potential:

$$V(k_0) = V_0(k_0)+V_1(k_0)+V_2(k_0)+ \ldots, \quad (25)$$

where the part $V_0$ coming from the uncorrelated density $\rho_0(\mathbf{r}_a, \mathbf{r}_b)$ is

$$V_0(k_0) = 4\int^{k_F} d\boldsymbol{\varkappa}\, l\left(\tfrac{1}{2}(\mathbf{k}_0 - \boldsymbol{\varkappa})\right). \quad (26)$$

By Fourier transforming the correction terms $\rho_1(\mathbf{r}_a, \mathbf{r}_b)$ and $\rho_2(\mathbf{r}_a, \mathbf{r}_b)$ and performing the integrations indicated in eq. (9) [or eq. (14)] we then find the respective contributions to the optical potential:

$$V_1(k_0) = \frac{16}{(2\pi)^3}\int^{k_F} d\boldsymbol{\varkappa}_1\, d\boldsymbol{\varkappa}_2 \left\{\left(\frac{1}{2\pi}\right)^3 \int d\mathbf{k}_1\, \tilde{l}\left(\tfrac{1}{2}(\mathbf{k}_0-\mathbf{k}_1)\right)\left[|A(\mathbf{k}_1-\boldsymbol{\varkappa}_1)|^2 - \tfrac{1}{4}A^*(\mathbf{k}_1-\boldsymbol{\varkappa}_1)A(\mathbf{k}_1-\boldsymbol{\varkappa}_2) + 2\tilde{l}\left(\tfrac{1}{2}(\mathbf{k}_0-\boldsymbol{\varkappa}_1)\right) \mathrm{Re}\left[A(0)-\tfrac{1}{4}A(\boldsymbol{\varkappa}_1-\boldsymbol{\varkappa}_2)\right]\right\}, \tag{27}$$

where

$$A(\mathbf{p}) = \int d\mathbf{r}\, e^{-i\mathbf{p}\cdot\mathbf{r}}\lambda(\mathbf{r}), \tag{28}$$

and

$$V_2(k_0) = -\frac{16}{(2\pi)^3}\int^{k_F} d\boldsymbol{\varkappa}_2 d\boldsymbol{\varkappa}_3\, \tilde{l}\left(\tfrac{1}{2}(\mathbf{k}_0-\boldsymbol{\varkappa}_2)\right)\{H(0)-\tfrac{1}{4}H(\boldsymbol{\varkappa}_2-\boldsymbol{\varkappa}_3)\}, \tag{29}$$

where

$$H(\mathbf{p}) = \int d\mathbf{r}\, e^{-i\mathbf{p}\cdot\mathbf{r}}h(r). \tag{30}$$

It is rather straightforward to generalize these corrections to the case of the state dependent correlation functions corresponding to $\psi_0$ of eq. (5). Let $\psi_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{r}_1, \mathbf{r}_2)$ be a solution of the Bethe-Goldstone (BG) equation [15, 25]:

$$(t_1+t_2-E_{12})\psi_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2} = -Q^F_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}V_{12}\psi_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2} \tag{31}$$

where $Q^F_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}$ is a projection operator which rejects all Fourier components that are not outside the Fermi distribution except the ones corresponding to the levels $\boldsymbol{\varkappa}_1$ and $\boldsymbol{\varkappa}_2$, and $t_1$ and $t_2$ are the kinetic energy operators which can include the effect of the momentum dependent one-particle potential e.g. in the form of an effective mass.

Let us introduce the function

$$\begin{aligned}\Delta\psi_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{r}_1, \mathbf{r}_2) &= \psi_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{r}_1, \mathbf{r}_2) - \varphi_{\boldsymbol{\varkappa}_1}(\mathbf{r}_1)\varphi_{\boldsymbol{\varkappa}_2}(\mathbf{r}_2) \\ &= \lambda_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(r_{12})\varphi_{\boldsymbol{\varkappa}_1}(\mathbf{r}_1)\varphi_{\boldsymbol{\varkappa}_2}(\mathbf{r}_2),\end{aligned} \tag{32}$$

which is a generalization of $\lambda(r_{12})\varphi_{\boldsymbol{\varkappa}_1}(\mathbf{r}_1)\varphi_{\boldsymbol{\varkappa}_2}(\mathbf{r}_2)$ of the previous case. We can write the Fourier transform of $\Delta\psi$ in the form:

$$\begin{aligned}\Delta\tilde{\psi}_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{k}_1, \mathbf{k}_2) &= \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{r}_1 d\mathbf{r}_2 e^{-i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)}\Delta\psi_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{r}_1, \mathbf{r}_2) \\ &= \frac{1}{\Omega}\delta(\mathbf{k}_1+\mathbf{k}_2-\boldsymbol{\varkappa}_1-\boldsymbol{\varkappa}_2)\int d\boldsymbol{\varrho} e^{-\frac{1}{2}i(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\varrho}}\lambda_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\rho)e^{\frac{1}{2}i(\boldsymbol{\varkappa}_1-\boldsymbol{\varkappa}_2)\cdot\boldsymbol{\varrho}} \\ &= \frac{1}{\Omega}\delta(\mathbf{k}_1+\mathbf{k}_2-\boldsymbol{\varkappa}_1-\boldsymbol{\varkappa}_2)A_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{k}_1-\boldsymbol{\varkappa}_1),\end{aligned} \tag{33}$$

where $A$ is defined in eq. (28). It may be easily shown [21] that in consequence of the BG equation we have

$$\Delta\tilde{\psi}_{\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2}(\mathbf{k}_1, \mathbf{k}_2) = 0 \quad \text{for} \quad k_1 < k_F, k_2 < k_F, \tag{34}$$

and even more, the Fourier transform of $\Delta\psi$ with respect to one particle only vanishes for momenta inside the Fermi sea. Let us notice that according to

eq. (33) we may write eq. (34) in the form

$$\Lambda_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)=0 \quad \text{for} \quad k_1<k_F. \tag{35}$$

As a result of these properties the terms linear in $\Lambda$ drop off in the expression for $V_1$ and $V_2$ and the integration over $\mathbf{k}_1$ in eq. (27) reduces to the region outside the Fermi sea. Our previous eqs. (27) and (28) go over into

$$V_1(k_0)=\frac{16}{(2\pi)^3}\int^{k_F} d\varkappa_1 d\varkappa_2 \Big\{\frac{1}{(2\pi)^3}\int_{k_F} d\mathbf{k}_1 \bar{t}\left(\tfrac{1}{2}(\mathbf{k}_0-\mathbf{k}_1)\right) \times\left[|\Lambda_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)|^2-\tfrac{1}{4}\Lambda^*_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)\Lambda_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_2)\right]\Big\} \tag{36}$$

$$V_2(\kappa_0)=-\frac{16}{(2\pi)^3}\int^{k_F} d\varkappa_2 d\varkappa_3 \bar{t}\left(\tfrac{1}{2}(\mathbf{k}_0-\varkappa_2)\right)\{L_{\varkappa_2\varkappa_3}(0)-\tfrac{1}{4}L_{\varkappa_2\varkappa_3}(\varkappa_2-\varkappa_3)\}, \tag{37}$$

where

$$L_{\varkappa_2\varkappa_3}(\mathbf{p})=\int d\mathbf{r}\, e^{-i\mathbf{p}\cdot\mathbf{r}}|\lambda_{\varkappa_2\varkappa_3}(\mathbf{r})|^2. \tag{38}$$

Let us notice that apart from the state dependence of the correlation functions the only real difference between eqs. (27), (29) and eqs. (36), (37) lies in the changed integration domain of $\mathbf{k}_1$ in eq. (36). In fact, one may easily see that the terms linear in $\Lambda$ would cancel anyhow while taking the sum $V_1+V_2$.

Since the nucleon-nucleon interaction $V_{12}$ in the BG eq. (31) depends on the spin and parity of the state of the two nucleons, the correlation function $\lambda$ also, in the general case, depends on the spin and parity of the two nucleons concerned. Let us neglect the difference between the interaction in odd and even states. In this case the function $\lambda$ depends on the spin only and we have to distinguish only between $\lambda^t$ and $\lambda^s$ for the triplet and singlet states of the two correlated nucleons. In this case one has to generalize the eqs. (36, 37) by substituting

$$\begin{aligned}
&|\Lambda_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)|^2 \to \tfrac{3}{4}|\Lambda^t_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)|^2+\tfrac{1}{4}|\Lambda^s_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)|^2\\
&\Lambda^*_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)\Lambda_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_2) \to \tfrac{3}{2}\Lambda^{t*}_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)\Lambda^t_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_2)\\
&\qquad -\tfrac{1}{2}\Lambda^{s*}_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_1)\Lambda^s_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\varkappa_2),\\
&L_{\varkappa_2\varkappa_3}(0) \to \tfrac{3}{4}L_{\varkappa_1:\ 3}(0)+\tfrac{1}{4}L^s_{\varkappa_2\varkappa_3}(0),\\
&L_{\varkappa_2\varkappa_3}(\varkappa_2-\varkappa_3) \to \tfrac{3}{2}L^t_{\varkappa_2\varkappa_3}(\varkappa_2-\varkappa_3)-\tfrac{1}{2}L^s_{\varkappa_2\varkappa_3}(\varkappa_2-\varkappa_3),
\end{aligned} \tag{39}$$

where the Fourier transforms $\Lambda^t, \ldots, L^s$ are defined by equations identical with eqs. (28), (38) with $\lambda$ replaced by $\lambda^t, \ldots, \lambda^s$ respectively.

The part $V_1$ of the optical potential corresponds to the diffuseness of the Fermi distribution produced by the correlation of the struck particle 1 with another target particle 2 as indicated by the form of the correction $\rho_1$ to the density $\rho$.

The form of $\rho_2$ shows that the correlations between the target particles other than the struck particle 1 (e.g. particles 2 and 3) affect the momentum

distribution of particle 1. In fact, the completely anti-symmetric nuclear wave function $\psi_0$ contains exchanges of the particle 1 with the particles 2 and 3. Accordingly, the part $V_2$ represents the effect of the exchange of the struck particle with one of the two other mutually correlated target particles.

To be able to proceed with the evaluation of $V_0$, $V_1$ and $V_2$ we must first calculate the $t$-matrix. To do it we must solve the integral equation (2) for a given form of the nucleon-nucleon interaction.

First of all we simplify this equation in the sense that we neglect in the intermediate states the modification of the Fermi distribution caused by the correlations. That means we use for the projection operator $Q$ in eq. (3) the approximate form

$$Q \approx Q^{\mathrm{F}}(\mathbf{k}_1, \mathbf{k}_2) = \begin{cases} 1 & \text{for} \quad k_1 > k_{\mathrm{F}} \quad \text{and} \quad k_2 > k_{\mathrm{F}}, \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

This approximation seems to be especially good at higher energies where the tail of the modified momentum distribution is very small. A further discussion of the approximation (40) is given in appendix 1.

The next point is the choice of the two-body interaction. The use of the most realistic two-body interaction reproducing in the best way all the two-body data would force us to apply a suitable approximation procedure which will be the subject of the next paper. At the moment let us satisfy ourselves with a simplified interaction which, however, would enable us to solve comparatively easily the eq. (2) for the $t$-matrix. An interaction of this type, which will be used in the following calculations, is the separable Yamaguchi potential [9]) of the form

$$\begin{aligned} \frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{r}_{12}\, \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{r}_{12}} V_{12}\, \mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}_{12}} &= \langle \mathbf{p}' | V_{12} | \mathbf{p} \rangle \\ &= -\frac{\lambda \hbar^2}{M} g(p') g(p), \end{aligned} \tag{41}$$

where

$$g(p) = (p^2 + \beta^2)^{-1} \tag{42}$$

with $\lambda = 0.414 f^{-3} (0.332 f^{-3})$ for the triplet (singlet) states and with $\beta = 1.449 f^{-1}$ for both the triplet and singlet states †.

The average self-consistent single particle potential $A\, t_{c0i}$ in the energy denominator is approximated by a quadratic function of momentum, i.e. we use the effective mass ($M^*$) approximation. According to the results of Verlet and Gavoret [8]) we use the value $M^* = 0.8\, M$. However, it is not quite clear if it is legitimate to use the quadratic approximation for the high momenta. In a realistic theory the effective mass is a function of the energy approaching the value of the ordinary mass $M$ in the high energy

† We use the length $f = 10^{-13}$ cm. The value of $\lambda$ for the singlet state differs slightly from the value 0.291 suggested by Yamaguchi [9]). With our choice we avoid some of the numerical calculations for the case $M^* = 0.8\, M$.

limit. For these reasons we perform our calculations also for $M^* = M$. Certainly, the correct result should lie between these two limits. Fortunately, as we shall see, the final results are not drastically sensitive to the change of $M^*$ from $0.8M$ to $M$.

For the interaction (41) the solution of eq. (2) with the effective mass approximation has the simple form

$$\langle \mathbf{p}'|t|\mathbf{p}\rangle = \frac{\langle \mathbf{p}'|V_{01}|\mathbf{p}\rangle}{1-\frac{M^*}{\hbar^2}\int \mathrm{d}\mathbf{k} Q(\frac{1}{2}\mathbf{K}+\mathbf{k}, \frac{1}{2}\mathbf{K}-\mathbf{k})\langle \mathbf{k}|V_{01}|\mathbf{k}\rangle/[p^2-k^2+i\eta]} \tag{43}$$

where $\mathbf{K}$ is the centre of mass momentum not indicated explicitly in our notation for $t$. The integral in the denominator may be easily expressed in a closed form with the help of elementary functions, although, because of the operator $Q$, the expression is rather long.

With the $t$-matrix given by eq. (43) we now may calculate $V_0$. This was done by Verlet and Gavoret [8]) and we shall simply use their numerical results.

To calculate $V_1$ and $V_2$ we still need a definite form of the correlation function. Strictly speaking, to determine the correlation function, we should solve the BG equation (31), which would be simple in principle if we used the interaction (41). Or otherwise, we could apply the variational principle to find the correlation function (cf. refs. [19, 20, 23])). Since, however, our problem is to estimate correction terms only to the main term $V_0$, we shall proceed in a simpler way: we shall use a simple analytical form given by Amado [26]) which fairly well reproduces the behaviour of the correlation function determined with the help of the BG equation. This form is

$$\lambda(r) = -\mathrm{e}^{-\gamma r^2}, \tag{44}$$

where $\gamma = 1.25f^{-2}$. The fact that $\lambda$ in eq. (44) does not depend on the state of the two correlated nucleons allows us to use the simpler expressions for $V_1$ and $V_2$ given by eqs. (27) and (29). Since $\lambda$ given by eq. (44) represents a solution of the BG equation we have to drop the terms linear in $\Lambda$, and we may write $V_1$ and $V_2$ in the form

$$V_1(k_0) = \rho_0{}^2 \int_{k_F} \mathrm{d}\mathbf{k}_1 \tilde{t}(\tfrac{1}{2}(\mathbf{k}_0-\mathbf{k}_1)) F(k_1) \tag{45}$$

where

$$F(k_1) = [|\Lambda(\mathbf{k}_1-\boldsymbol{\varkappa}_1)|^2 - \tfrac{1}{4}\Lambda^*(\mathbf{k}_1-\boldsymbol{\varkappa}_1)\Lambda(\mathbf{k}_1-\boldsymbol{\varkappa}_2)]_{\mathrm{Av}} \tag{46}$$

where $[]_{\mathrm{Av}}$ means an average value over the Fermi sea, and

$$V_2(k_0) = -4\rho_0 \xi \tfrac{3}{4} L(0) \int^{k_F} \mathrm{d}\boldsymbol{\varkappa}_2 \tilde{t}(\tfrac{1}{2}(\mathbf{k}_0-\boldsymbol{\varkappa}_2)). \tag{47}$$

If we neglected in eq. (47) the dependence of the exchange term on $\boldsymbol{\varkappa}_2$ and $\boldsymbol{\varkappa}_3$, i.e. if we put $L(\boldsymbol{\varkappa}_2-\boldsymbol{\varkappa}_3) \approx L(0)$ we would have $\xi = 1$. To calculate $\xi$ the

exchange term $L(\varkappa_2-\varkappa_3)$ was averaged over the Fermi sea and the value 1.05 for $\xi$ was obtained.

By comparison with eq. (26) we see that

$$V_2(k_0) = -\tfrac{3}{4}\rho_0\xi L(0)V_0(k_0) = -\tfrac{3}{4}\rho_0\xi\left(\frac{\pi}{2\gamma}\right)^{\frac{3}{2}}V_0(k_0). \tag{48}$$

This means that the relative correction to $V_0$ introduced by $V_2$ is constant and equals $-\frac{3}{4}\xi\rho_0 L(0)$.

As we saw before, if we did not use a correlation function which satisfies the BG equation, we should only change the integration region $k_1 > k_F$ in eq. (45) into an integration of $\mathbf{k}_1$ over the whole momentum space. In our numerical calculations we shall consider the two possibilities for the integration region of $\mathbf{k}_1$. This seems to be advisable since the simple form of $\lambda$, eq. (44) is, of course, not an exact solution of the BG equation.

While performing the averaging over the Fermi sea indicated in eq. (46) we neglect the cosines of the angle between $\varkappa_{1(2)}$ and $\mathbf{k}_1$ and obtain

$$\rho_0{}^2F(k_1) = 0.116\,\mathrm{e}^{-k_1{}^2/2\gamma}, \tag{49}$$

where the value $k_F = 1.2\,f^{-1}$ was used.

To simplify the calculation further we shall use the free scattering $t_0$-matrix, i.e. we shall neglect the projection operator $Q$ in eq. (43). To justify this approximation we must remember that we are calculating here only a correction to the optical potential. On the other hand, this is certainly a very reasonable approximation at higher energies, where the fact that the low energy states below the Fermi level are occupied becomes less important.

With this approximation our $t_0$-matrix depends on $\frac{1}{2}|\mathbf{k}_0-\mathbf{k}_1| = p$ only, and we have

$$V_1(k_0) = \rho_0{}^2\int \mathrm{d}\mathbf{k}_1\tilde{t}_0(p)F(k_1) = \rho_0{}^2\frac{8\pi}{k_0}\int_0^\infty \mathrm{d}p\; pt_0(p)\int_{k_0-2p}^{k_0+2p}\mathrm{d}k_1 k_1 F(k_1). \tag{50}$$

If we introduce the restriction $k_1 > k_F$, then for $\frac{1}{2}(k_0-k_F) < p < \frac{1}{2}(k_0+k_F)$ one has to change the lower limit of the integration over $k_1$ in eq. (50) $k_0-2p$ into $k_F$. The integration over $k_1$ is elementary. The integration over $p$ was performed graphically.

Because of the approximation applied in the calculation of $V_1$, it seems to be advisable to perform the calculation of $V_1$ once more with the help of another form of the correlation function. Let us confine ourselves to the case in which we integrate over $k_1 > k_F$ [eqs. (36), (45)]. In this case one may rewrite $F(k_1)$ of eq. (46) in the form

$$\begin{aligned} F(k_1) = [&|\tilde{\psi}_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\tfrac{1}{2}(\varkappa_1+\varkappa_2))|^2-\tfrac{1}{4}\tilde{\psi}^*_{\varkappa_1\varkappa_2}(\mathbf{k}_1-\tfrac{1}{2}(\varkappa_1+\varkappa_2)) \\ &\times\tilde{\psi}_{\varkappa_2\varkappa_1}(\mathbf{k}_1-\tfrac{1}{2}(\varkappa_1+\varkappa_2))]_{\mathrm{Av}}, \end{aligned} \tag{51}$$

where

$$\psi_{\varkappa_1\varkappa_2}(\mathbf{r}) = [1+\lambda_{\varkappa_1\varkappa_2}(\mathbf{r})]\mathrm{e}^{\mathrm{i}\frac{1}{2}(\varkappa_1-\varkappa_2)\cdot\mathbf{r}}, \tag{52}$$

and

$$\tilde{\psi}_{\varkappa_1\varkappa_2}(\mathbf{p}) = \int \mathrm{d}\mathbf{r}\, \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{r}} \psi_{\varkappa_1\varkappa_2}(\mathbf{r}). \tag{53}$$

Since $k_1 > k_F > \kappa_{1(2)}$ we shall here neglect the influence of the other particles in the Fermi sea and use for $\tilde{\psi}_{\varkappa_1\varkappa_2}$ the Fourier transform of the "quasi-deuteron" wave function used by Heidman [11]) and Levinger [12]). Following Heidman we use for $\psi_{\varkappa_1\varkappa_2}$ the triplet wave function for the S state:

$$\psi_{\varkappa_1\varkappa_2}(\mathbf{r}) \approx \{\mathrm{e}^{-\alpha r} - \mathrm{e}^{-\mu r}\} \frac{1}{r} \frac{1}{(\alpha^2+\kappa^2)^{\frac{1}{2}}}, \tag{54}$$

where $\varkappa = \frac{1}{2}(\varkappa_1 - \varkappa_2)$, $\alpha = 0.186\, f^{-1}$, $\mu = 0.847\, f^{-1}$. Using the result of Levinger [12])

$$\left[\frac{1}{\alpha^2+\kappa^2}\right]_{\mathrm{Av}} \approx \frac{4}{k_F{}^2}, \tag{55}$$

and neglecting $\frac{1}{2}(\varkappa_1+\varkappa_2)$ as compared with $\mathbf{k}_1$ in the argument of the Fourier transforms $\tilde{\psi}_{\varkappa_1\varkappa_2}$, we get

$$F(k_1) = \tfrac{3}{4}(4\pi)^2 \left\{\frac{\mu^2-\alpha^2}{(k_1{}^2+\mu^2)(k_1{}^2+\alpha^2)}\right\}^2 \frac{4}{k_F{}^2}. \tag{56}$$

This form of $F(k_1)$ inserted into eq. (50) [with the restriction $k_1 > k_F$] supplies us with $V_1$ corresponding to the "quasi deuteron" correlation function.

The numerical results obtained for $V_1$ and $V_2$, together with the results of Verlet and Gavoret for $V_0$, are presented and discussed in the next section.

## 3. Numerical Results and Their Interpretation

In all the numerical calculation we assume the nuclear density corresponding to the value of $1.27f$ for $r_0 = (\frac{3}{4}\pi\rho_0)^{\frac{1}{3}}(k_F = 1.2f^{-1})$ and the energy of the top of the Fermi sea $\Delta E = -10$ MeV. This choice enables us to use the values of $V_0$ given in ref. [8]).

We will discuss the optical potential as a function of $k_0$, the wave number of the incoming nucleon. The energy $E_0$ of the incoming nucleon is connected with $k_0$ through the equation:

$$E_0 = \frac{\hbar^2 k_0{}^2}{2M^*} - \frac{\hbar^2 k_F{}^2}{2M^*} + \Delta E. \tag{57}$$

Since the Coulomb effect is not taken into account in this paper the results obtained are applicable for an incoming neutron rather than a proton except at the highest energies considered.

The part $V_2$ of the optical potential calculated according to eq. (48) gives a $-13$ per cent correction to $V_0$:

$$V_2(k_0) = -0.13\, V_0(k_0). \tag{58}$$

The constant factor is universal, i.e. independent of the energy and the effective mass. If one uses the second form of the correlation function given in ref. [26]), namely, the Yukawa form

$$\lambda(r) = \begin{cases} 1 - \dfrac{r_c}{r} e^{-\nu(r-r_c)}, & r > r_c, \\ 0 \qquad\qquad\quad , & r < r_c, \end{cases} \tag{59}$$

with $r_c = 0.4f$ and $\nu = 0.87^{-1}$, the factor 0.13 in eq. (58) would be reduced by approximately 18 per cent.

The potential $V_1$ was computed for two cases: A) with the BG condition (integration over $k_1 > k_F$ in eq. (45)), and B) without this condition (integration over all the values of $k_1$).

The results of the computation in Case A) are given in table 1 for the two forms of the correlation function: the gaussian function of Amado [26]) and

TABLE 1

Values of $V_1$ corresponding to the "quasi deuteron" and the gaussian correlation functions, and the B–G condition taken into account. All energies are given in MeV.

| $k_0/k_F$ | | 1 | 1.5 | 2 | 3 | correlation function |
|---|---|---|---|---|---|---|
| $M^* = 0.8M$ | $E_0$ | −10 | +27 | +79 | +227 | |
| | Re $V_1$ | − 1.04<br>− 1.53 | − 0.94<br>− 1.36 | − 0.78<br>− 1.15 | − 0.42<br>− 0.72 | quasi deut.<br>gaussian |
| | Im $V_1$ | − 1.92<br>− 2.09 | − 1.30<br>− 1.85 | − 0.74<br>− 1.23 | − 0.21<br>− 0.42 | quasi deut.<br>gaussian |
| $M^* = M$ | $E_0$ | −10 | 36 | 101 | 286 | |
| | Re $V_1$ | − 0.67<br>− 1.08 | − 0.65<br>− 0.96 | − 0.61<br>− 0.87 | − 0.35<br>− 0.59 | quasi deut.<br>gaussian |
| | Im $V_1$ | − 1.61<br>− 1.81 | − 1.14<br>− 1.58 | − 0.67<br>− 1.09 | − 0.20<br>− 0.39 | quasi deut.<br>gaussian |

the "quasi-deuteron" function of ref. [11]) and [12]). The cases of $M^* = 0.8M$ and $M^* = M$ are considered, the latter being probably more realistic for the two highest energies, in accordance with the remarks in the previous section. In the case $M^* = 0.8\,M$ we simplified the calculation by assuming that the interaction in singlet states is the same as that in triplet states. Four incoming neutron energies are used, corresponding to $k_0/k_F = 1$, 1.5, 2 and 3. The two first cases are not to be considered seriously as our approximations

† For $M^* = 0.8M$ the uncorrelated $V_0$ in ref.[8]) has been calculated with the "effective range" separable potential. This form of the potential, however, gives results very close to those given by the form (42), as is shown in ref.[8]) in the case $M^* = M$.

are invalid in the low energy range; they are merely shown to indicate the general trend of the energy dependence of $V_1$. This remark applies also to all the other low energy results presented in this section.

In table 2 the relative contributions of $\Delta V = V_1+V_2$ are presented with the numbers of table 1 for the Amado correlation function in form of the ratios (in per cent) of $\Delta V$ to $V_0$ the "uncorrelated" (main part) of the optical potential taken from the paper by Verlet and Gavoret [8]).

TABLE 2

The ratios in per cent $\Delta V = V_1+V_2$ for the gaussian correlation function with the B–G condition to the uncorrelated optical potential $V_0$ taken from ref.[8]).

| $k_0/k_F$ | | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|---|
| $M^* = 0.8M$ | $\frac{\mathrm{Re}\,\Delta V}{\mathrm{Re}\,V_0}$ | | −10.4 | −9.4 | −10.2 |
| | $\frac{\mathrm{Im}\,\Delta V}{\mathrm{Re}\,V_0}$ | $+\infty$ | + 5.5 | −3.5 | − 7.4 |
| $M^* = M$ | $\frac{\mathrm{Re}\,\Delta V}{\mathrm{Re}\,V_0}$ | −11.4 | −11.0 | −9.2 | − 6.2 |
| | $\frac{\mathrm{Im}\,\Delta V}{\mathrm{Im}\,V_0}$ | $+\infty$ | − 5.3 | −4.0 | − 0.3 |

The final results of the computations in the case B) are presented in table 3. The $+\infty$ corresponds to the fact that for the "incoming energy" $=-10$ MeV the "uncorrelated" $V_0$ has no imaginary part while $\mathrm{Im}\,V_1 \neq 0$, as the correlations wash out the momentum distribution of the Fermi gas and we can have absorptions even for $k_0 < k_F$.

TABLE 3

The ratios in per cent of $\Delta V = V_1+V_2$ for the gaussian correlation function without the B–G condition to the uncorrelated optical potential $V_0$ taken from ref. [8]).

| $k_0/k_F$ | | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|---|
| $M^* = 0.8M$ | $\frac{\mathrm{Re}\,\Delta V}{\mathrm{Re}\,V_0}$ | | − 9.2 | −7.5 | −7.2 |
| | $\frac{\mathrm{Im}\,\Delta V}{\mathrm{Im}\,V_0}$ | $+\infty$ | +12.3 | +0.2 | −6.3 |
| $M^* = M$ | $\frac{\mathrm{Re}\,\Delta V}{\mathrm{Re}\,V_0}$ | −10.7 | −10.1 | −7.2 | −4.2 |
| | $\frac{\mathrm{Im}\,\Delta V}{\mathrm{Im}\,V_0}$ | $+\infty$ | − 2.0 | −0.3 | +2.4 |

From table 1 we see that $V_1$ is not very sensitive to the choice of the correlation function $\lambda(r)$ and that both its real and imaginary parts are of the order

of 1 MeV for all the energies considered. The ratios of the sum $V_1+V_2 = \Delta V$ are also rather small and hardly exceeding $-10$ per cent of Re $V_0$ for the real part and $\pm 5$ per cent of Im $V_0$ for the imaginary part as seen from tables 2 and 3. The BG condition does not change the results very much: it only decreases both the real and the imaginary part of $V_1$. This indicates that the Amado correlation function does not deviate very much from an exact solution of the BG equation at high energies.

The contribution of $V_2$ is relatively big and of opposite sign with respect to $V_0$ and $V_1$. The two contributions $V_1$ and $V_2$ fight each other reducing the global effect of correlations.

From tables 2 and 3 we see that for $M^* = 0.8M$ the resulting imaginary part of the optical potential is increased by the correlations at medium energies and is slightly diminished only at the highest energies. That means that Im $V_1$ dominates over Im $V_2$ and only at very high energies $V_2$ takes over. Re $V_2$ dominates over Re $V_1$ for all the cases in table 2 and 3, and as a result the real part of the optical potential is decreased by the correlation effects. The total effect of Im $\Delta V$ = Im$(V_1+V_2)$ is seen to be sensitive to the effective mass, and for $M^* = M$ it is of opposite sign to that for $M^* = 0.8M$ in an appreciable energy range, i.e. here Im $\Delta V$ decreases the well depth of the imaginary part of the optical potential.

Brueckner [7]) computed the imaginary part of the optical potential at very low energies using a slightly modified form of the "frivolous" model, replacing the $t$-matrix by a $K$-matrix with the neglect of the centre of mass momentum. He only took into account one particular correction corresponding to correlations: the departure of the nuclear momentum distribution from the Fermi gas in computing the effects of the Pauli principle in the intermediate states. He found this effect to increase slightly the imaginary well depth. This particular effect is discussed in our appendix 1: it clearly becomes less and less important the higher the projectile energy and should be negligible at energies as high as $k_0 > 2k_F$ or so.

The effect of our $V_1$ acts in the same direction of increasing the imaginary well depth. On the other hand $V_2$, describing the effects of the exchanges among the strongly correlated target particles, acts in the opposite direction.

In view of so many approximations involved, our results should be considered as rather of a qualitative character. Of all the effects neglected up to now, the most important seems to be the effect of the incoherent multiple scattering of the projectile with two subsequent target particles correlated with each other.

## 4. Corrections of Incoherent Scattering

The first correction term of the incoherent multiple scattering results from the first iteration of eq. (4) for $G_{0i}$:

$$G_{0i} \approx 1+\frac{Q}{e}\sum_{j\neq i} I_{0j}. \tag{60}$$

The resulting correction term to the optical potential is

$$\langle \mathbf{k}'_0|V_{(1n)}|\mathbf{k}_0\rangle = (1/2\pi)^3 \langle e^{i\mathbf{k}'_0\cdot\mathbf{r}_0}\psi_0| \sum_{j\neq i} I_{0i}\frac{Q}{e}I_{0j}|e^{i\mathbf{k}_0\cdot\mathbf{r}_0}\psi_0\rangle, \tag{61}$$

where the initial and final states should be properly antisymmetrized in all the respective pairs of interacting particles. The matrix element of eq. (61) involves only transitions of the target nucleus from its ground state $\psi_0$ to excited states and then back again to the ground state. No single particle intermediate state can be identical with any of the states present in the ground state. Eq. (61) describes an inelastic scattering by one target nucleon followed by a second inelastic scattering by another target nucleon which brings the target nucleus back to its original state.

If one is not strict about the $Q$-operator and replaces $\langle I_{0i}\frac{Q}{e}I_{0j}\rangle$ of eq. (61) by

$$\langle I_{0i}\frac{Q}{e}I_{0j}\rangle \to \langle t_{0i}\frac{1}{e}t_{0j}\rangle - \langle t_{0i}\rangle\frac{1}{e}\langle t_{0j}\rangle, \tag{62}$$

one can show that in the high energy limit

$$\langle \mathbf{k}'_0|V_{(1n)}|\mathbf{k}_0\rangle \to A(A-1)\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{p}_0 [\rho(\mathbf{r}_1\mathbf{r}_2)-\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)] \times e^{i(\mathbf{p}_0-\mathbf{k}'_0)\mathbf{r}-i(\mathbf{p}_0-\mathbf{k}_0)\cdot\mathbf{r}_2}\frac{\langle\frac{1}{2}\mathbf{k}'_0|t|\frac{1}{2}\mathbf{p}_0\rangle\langle\frac{1}{2}\mathbf{p}_0|t|\frac{1}{2}\mathbf{k}_0\rangle}{\frac{\hbar^2}{2M^*}[k_0^2-p_0^2+i\eta]}. \tag{63}$$

where $\rho(\mathbf{r}_1, \mathbf{r}_2)$ is a two-particle density.

Eq. (63) is of the form known from Watson's theory [2]) and also discussed by Feshbach [27]). One must be careful, however, in applying eq. (63) to the wave functions of nuclear matter of the type used in the present paper. In fact, if we retain only terms up to the order $h(r_{12})$ in the expansion of $[\rho(\mathbf{r}_1, \mathbf{r}_2)-\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)]$ we get

$$[\rho(\mathbf{r}_1, \mathbf{r}_2)-\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)] \approx \rho_0^2[h(r_{12})\{1-\tfrac{1}{4}l^2(r_{12})\}-\tfrac{1}{4}l^2(r_{12})]. \tag{64}$$

This means that even in the absence of any dynamical correlations, i.e., if $h \to 0$ (in the case of the pure Fermi gas), we would have a non-vanishing contribution of $V_{(1n)}$ from the statistical correlations introduced by the Pauli principle alone.

Let us be strict and re-derive $V_{(1n)}$ using the $Q$-operator properly. The initial and final states are antisymmetrized in the coordinates of the particles interacting in the first and the second scattering respectively. Then one may write eq. (61) in the form

$$\langle \mathbf{k}'_0|V_{(1n)}|\mathbf{k}_0\rangle = A(A-1)(2\pi)^{-3}\int d(0)d(1)\ldots d(A)\{(1-P_{01})e^{-i\mathbf{k}'_0\cdot\mathbf{r}_0}\chi^*_{s_0}(\sigma_0) \times \eta^*_{t_0}(\tau_0)\psi_0^*(1\ldots A)\}t_{01}\frac{Q}{e}t_{02}(1-P_{02})e^{i\mathbf{k}_0\cdot\mathbf{r}_0}\chi_{s_0}(\sigma_0)\eta_{t_0}(\tau_0)\psi_0(1\ldots A), \tag{65}$$

where $1-P_{0i} = 1-P^{r}_{0i}P^{\sigma}_{0i}P^{\tau}_{0i}$ is an operator antisymmetrizing the wave function in the spatial, spin and isospin coordinates of the incoming particle 0 and the target particle $i$.

Our first assumption is that $t_{01}\frac{Q}{e}t_{02}$ does not act on the spin and isospin functions. This is certainly the case if we assume the same form (41) of the nucleon-nucleon interaction in the triplet and singlet states. This is to simplify the derivation — the generalization is straightforward.

There are four amplitudes involved in $V_{(\ln)}$ of eq. (65). Let us now proceed to evaluate the direct-direct amplitude $A_{\mathrm{D}}$ of eq. (65). Let us now use plane waves for the intermediate states and replace the operator $Q$ by $Q^{\mathrm{F}}$ as we have done before, eq. (40), and discussed there. One may show that the amplitude $A_{\mathrm{D}}$ is then approximately

$$A_{\mathrm{D}} \approx A(A-1)\frac{2M^*}{\hbar^2}\int_{k_F}^{\infty}\mathrm{d}\mathbf{p}_0\,\frac{\langle\tfrac{1}{2}\mathbf{k}'_0|t|\tfrac{1}{2}\mathbf{p}_0\rangle\langle\tfrac{1}{2}\mathbf{p}_0|t|\tfrac{1}{2}\mathbf{k}_0\rangle}{k_0^2-p_0^2+i\eta} \times\int_{k_F}^{\infty}\mathrm{d}\varkappa_1\int_{k_F}^{\infty}\mathrm{d}\varkappa'_2\,\rho(\mathbf{p}_0+\varkappa_1-\mathbf{k}_0',\,\varkappa'_2;\,\varkappa_1,\,\mathbf{p}_0+\varkappa'_2-\mathbf{k}_0), \tag{66}$$

where

$$\rho(\varkappa'_1\varkappa'_2;\,\varkappa_1\varkappa_2) = \left(\frac{1}{2\pi}\right)^6\int\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\mathrm{d}\mathbf{r}'_1\mathrm{d}\mathbf{r}'_2\mathrm{e}^{i(\varkappa'_1\cdot\mathbf{r}'_1+\varkappa'_2\cdot\mathbf{r}'_2)}\rho(\mathbf{r}'_1\,\mathbf{r}'_2;\,\mathbf{r}_1\mathbf{r}_2) \times\mathrm{e}^{-i(\varkappa_1\cdot\mathbf{r}_1+\varkappa_2\cdot\mathbf{r}_2)}, \tag{67}$$

and

$$\rho(\mathbf{r}'_1\mathbf{r}'_2;\,\mathbf{r}_1\mathbf{r}_2) = \int\mathrm{d}\sigma_1\mathrm{d}\tau_1\mathrm{d}\sigma_2\mathrm{d}\tau_2\int\mathrm{d}(3)\ldots\mathrm{d}(A)\psi_0^*(\mathbf{r}'_1\sigma_1\tau_1,\,\mathbf{r}_2\sigma_2\tau_2,\,3\ldots A) \times\psi_0(\mathbf{r}_1\sigma_1\tau_1,\,\mathbf{r}_2\sigma_2\tau_2,\,3\ldots A). \tag{68}$$

In obtaining eq. (66) a very crude approximation was made as neglecting the momenta of the target particles 1 and 2 in the $\langle|t|\rangle$-amplitudes and in the energy denominator as compared with the momenta of the projectile 0. This approximation is valid only in the limit of very high energy scattering.

To proceed we evaluate first the mixed two particle density $\rho(\mathbf{r}'_1\mathbf{r}'_2;\,\mathbf{r}_1\mathbf{r}_2)$ using $\psi_0$ in the form given by eq. (5) with $\psi_{\varkappa_1\varkappa_2}$ of eq. (32) satisfying the BG equation, and keeping in the cluster development terms to the order $|\lambda_{\varkappa_\alpha\varkappa_\beta}(r_{12})|^2$. It turns out that in virtue of the BG equation the terms linear in $\lambda$ drop out and finally one arrives at

$$A_{\mathrm{D}} \approx \delta(\mathbf{k}'_0-\mathbf{k}_0)(2\pi)^3\frac{2M^*}{\hbar^2}\rho_0^2\int_{k_F}^{\infty}\mathrm{d}\mathbf{p}_0\int\mathrm{d}\mathbf{r}\mathrm{e}^{i(\mathbf{p}_0-\mathbf{k}_0)\cdot\mathbf{r}} \times|\lambda_{\varkappa_1\varkappa_2}(r)|^2_{\mathrm{Av}}[1-\tfrac{1}{4}l^2(r_{12})]\,\frac{|\langle\tfrac{1}{2}\mathbf{k}_0|t|\tfrac{1}{2}\mathbf{p}_0\rangle|^2}{k_0^2-p_0^2+i\eta}, \tag{66}$$

where $|\lambda_{\varkappa_1\varkappa_2}(r)|^2_{\mathrm{Av}}$ is the average of $|\lambda_{\varkappa_1\varkappa_2}(r)|^2$ over the Fermi sea with respect to $\varkappa_1$ and $\varkappa_2$.

Similarly one can compute all the exchange amplitudes. It turns out that finally, with a notation analogous to that introduced in eq. (13), we have

$$V_{(\mathrm{in})}(k_0) = (2\pi)^3 \frac{2M^*}{\hbar^2} \rho_0^2 \int_{k_F}^{\infty} \mathrm{d}\mathbf{p}_0 \int \mathrm{d}\mathbf{r} e^{i(\mathbf{p}_0-\mathbf{k}_0)\cdot\mathbf{r}} |\lambda_{\varkappa_1\varkappa_2}(r)|^2_{\mathrm{Av}} \times [1-\tfrac{1}{4}l^2(r)] \frac{|\langle \tfrac{1}{2}\mathbf{k}_0|t|\tfrac{1}{2}\mathbf{p}_0\rangle - \tfrac{1}{4}\langle -\tfrac{1}{2}\mathbf{k}_0|t|\tfrac{1}{2}\mathbf{p}_0\rangle|^2}{k_0^2 - p_0^2 + i\eta}. \tag{67}$$

We see that in this formulation the contribution to $V_{(\mathrm{in})}$ in the approximation (60) comes entirely from the dynamical correlation represented by $\lambda$, and that $V_{(\mathrm{in})}$ vanishes for the pure Fermi gas.

In evaluating the magnitude of $V_{(\mathrm{in})}$ we restrict ourselves to the imaginary part Im $V_{(\mathrm{in})}$ We find approximately

$$\mathrm{Im}\, V_{(\mathrm{in})}(k_0) \approx -\frac{M}{\hbar^2}\pi k_0 (2\pi)^3 \rho_0^2 \frac{1}{4\pi}\int \mathrm{d}\Omega_{p_0} \left\{ |\langle \tfrac{1}{2}\bar{\mathbf{k}}_0|t|\tfrac{1}{2}k_0 \frac{\mathbf{p}_0}{p}\rangle - \tfrac{1}{4}\langle -\tfrac{1}{2}\mathbf{k}_0|t|\tfrac{1}{2}k_0\frac{\mathbf{p}_0}{p_0}\rangle|^2 \right\} \times 4\pi\bar{\omega}_{ck_0}. \tag{68}$$

where the average "correlation volume"

$$\bar{\omega}_{ck_0} \equiv \frac{4\pi}{3} r^3_{ck_0} \equiv \tfrac{1}{2}\int_{-1}^{+1} \mathrm{d}x 4\pi \int_0^{\infty} r^2 \mathrm{d}r j_0(k_0\sqrt{2(1-x)}r) |\lambda_{\varkappa_1\varkappa_2}(r)|^2_{\mathrm{Av}} \times [1-\tfrac{1}{4}l^2(r)] \tag{69}$$

and

$$x = \cos(\mathbf{p}_0, \mathbf{k}_0).$$

If we confine ourselves to an S-state interaction for which we have $\langle -\tfrac{1}{2}\bar{\mathbf{k}}_0|t|\tfrac{1}{2}\bar{\mathbf{k}}_0\rangle = \langle \tfrac{1}{2}\bar{\mathbf{k}}_0|t|\tfrac{1}{2}\bar{\mathbf{k}}_0\rangle = t(\tfrac{1}{2}k_0)$; then we can rewrite eq. (68) in the form

$$\mathrm{Im}\, V_{(\mathrm{in})}(k_0) = -\frac{M^*}{\hbar^2}(\tfrac{3}{4})^2 \pi k_0 (2\pi)^3 \rho_0^2 |t(\tfrac{1}{2}k_0)|^2 4\pi\bar{\omega}_{ck_0}. \tag{70}$$

On the other hand the usual Fermi gas "frivolous" model gives the imaginary part of the optical potential as (28—29):

$$\mathrm{Im}\, \tilde{V}_0(k_0) = 4\int^{k_F} \mathrm{d}\bar{\mathbf{k}}_1 (-\tfrac{1}{2}\pi k_{01}) \frac{M^*}{\hbar^2}\int \mathrm{d}\Omega_p Q^{\mathrm{F}} \left| \left\langle \mathbf{k}_{01}|t_0|k_{01}\frac{\mathbf{p}}{p}\right\rangle \right|^2_{\mathrm{Av}} \to -(2\pi)^3\rho_0 \tfrac{1}{4}\pi k_0 \frac{M^*}{\hbar^2}\int \mathrm{d}\Omega_{p_0} \left| \left\langle \tfrac{1}{2}\mathbf{k}_0|t_0|\tfrac{1}{2}k_0\frac{\mathbf{p}_0}{p_0}\right\rangle \right|^2_{\mathrm{Av}} \tag{71}$$

in the very high energy limit $t_0$ stands for a free scattering matrix and the quantity $|\langle t_0\rangle|^2_{\mathrm{Av}}$ is proportional to $\frac{1}{2}(\sigma_{\mathrm{np}}(\theta)+\sigma_{\mathrm{nn}}(\theta))$ where the $\sigma(\theta)$'s are the elementary nucleon-nucleon scattering differential cross-sections. In case of a spin independent S-state interaction we can rewrite eq. (71) as

$$\mathrm{Im}\, \tilde{V}_0(k_0) = -(2\pi)^3\rho_0 \tfrac{1}{4}\pi k_0 4\pi \frac{M^*}{\hbar^2}|t_0(\tfrac{1}{2}k_0)|^2. \tag{72}$$

It follows immediately that the ratio of Im $V_{(\mathrm{in})}$ to Im $\tilde{V}_0$ in this limit is

$$R = \frac{\mathrm{Im}\, V_{(\mathrm{in})}(k_0)}{\mathrm{Im}\, \tilde{V}_0(k_0)} = \tfrac{9}{4}\left(\frac{r_{ck_0}}{r_0}\right)^3, \tag{73}$$

where $r_0 = (3/4\pi\rho_0)^{\frac{1}{3}}$, provided $t$ in eq. (70) is replaced by $t_0$.

The quantity $\bar{\omega}_{ck_0}$, and consequently $r_{ck_0}$, may be easily calculated with the gaussian form of the correlation function, eq. (44), especially if one approximates the factor $[1-\frac{1}{4}l^2(r)]$ simply by $\frac{3}{4}$. One then gets

$$\bar{\omega}_{ck_0} \approx \tfrac{3}{4}4\pi \left(\frac{\pi}{2\gamma}\right)^{\frac{3}{2}} \frac{2\gamma}{k_0^2} (1-e^{-k_0^2/2\gamma}). \tag{74}$$

Consequently, for $k_0 \to \infty$, the ratio $R$ decreases as $k_0^{-2}$. For the two highest energies considered in the previous section we find

$$R = \begin{cases} 0.11 & \text{for} \quad k_0 = 2k_F \\ 0.05 & \text{for} \quad k_0 = 3k_F. \end{cases}$$

On the other hand, with $\bar{\omega}_{ck_0}$ of eq. (72), we can calculate $\operatorname{Im} V_{(\mathrm{in})}(k_0)$ from eq. (68) using the separable interaction of eq. (41). We further simplify the calculation by setting the interaction in singlet state equal to that in triplet state and replace $t$ by $t_0$. We obtain

$$\operatorname{Im} V_{(\mathrm{in})}(k_0 = 2k_F) = \begin{cases} -2.3 \text{ MeV } (18\,\%) & \text{for} \quad M^* = 0.8M, \\ -2.5 \text{ MeV } (21\,\%) & \text{for} \quad M^* = M, \end{cases}$$

and

$$\operatorname{Im} V_{(\mathrm{in})}(k_0 = 3k_F) = \begin{cases} -0.3 \text{ MeV } (4\,\%) & \text{for} \quad M^* = 0.8M \\ -0.3 \text{ MeV } (10\,\%) & \text{for} \quad M^* = M, \end{cases}$$

where the numbers in the brackets give the ratio $\operatorname{Im} V_{(\mathrm{in})}/\operatorname{Im} V_0$.

Although our estimates of $V_{(\mathrm{in})}(k_0)$ are rather rough, we can conclude from the above results that the effect of incoherent scattering is important and of the same order of magnitude as the effects of $V_1(k_0)$ and $V_2(k_0)$ except for very high energies.

The relative contribution of $V_{(\mathrm{in})}(k_0)$ at high energies is expected to be inversely proportional to the laboratory energy of the scattering particle.

The authors feel indebted to Dr. L. Verlet for his valuable comments and information concerning his paper. One of the authors (J. S.) would like to acknowledge an interesting discussion with Dr. C. A. Levinson. One of us (J. D.) would like to express his gratitude to Professor R. E. Peierls for a valuable discussion and to Dr. N. M. Hughenholtz for his interesting comments. This work was accomplished while one of us (J. D.) was staying at the Institute for Advanced Study. It is his pleasure to thank Dr. J. R. Oppenheimer for his hospitality at the Institute. One of us (J. D.) was supported by a grant from the Institute for Advanced Study.

## Appendix 1

### EFFECT OF NON-DEGENERACY OF NUCLEAR GROUND STATE ON THE $t$-MATRIX

The effect of the exclusion principle on the intermediate states in the calculation of the $t$-operator has been taken into account in form corre-

sponding to a perfectly degenerate Fermi gas throughout the present paper. On the other hand the nuclear ground state wave function has been of the form including correlations which naturally introduce a non-degeneracy effect by "washing out" the Fermi momentum distribution. For a completely degenerate state the momentum distribution function is

$$P_{\mathrm{F}}(p)=\begin{cases}1, & p<k_{\mathrm{F}},\\ 0, & p>k_{\mathrm{F}}.\end{cases} \tag{A.1}$$

The corresponding function $P(p)$ for the correlated $\psi_0$ of the present paper was computed using the $\rho_1$ and $\rho_2$ density corrections:

$$P(p)=P_{\mathrm{F}}(p)+P_1(p)+P_2(p). \tag{A.2}$$

The resulting curves are shown in fig. 1: a) for the Amado gaussian correlation function without the BG condition; b) the same with the BG condition.

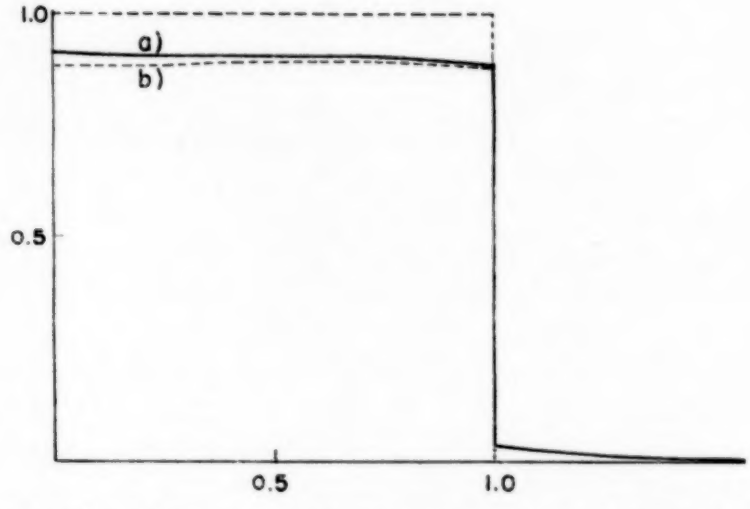


Fig. 1. Departure of the nuclear momentum distribution from a Fermi gas for the Amado gaussian function $\lambda(r)$: a) without the B–G condition; b) with the B–G condition.

It turns out that the most important effect comes from $\rho_2$, i.e. from $P_2(p)$. It cuts down the probability of occupation of states with $p<k_{\mathrm{F}}$ by about 10 per cent almost uniformly over the $(0, k_{\mathrm{F}})$ interval. As a compensation we get a small tail of higher momenta with $p>k_{\mathrm{F}}$.

This distribution is somewhat similar to that obtained by Brueckner [7]) (compare his fig. 4) by a different method. Brueckner then computed the imaginary well depth using the "frivolous" model modified by the integration with the use of the modified $P(p)$.

In our case the only effect to be considered is the modification of the integral

$$I_{\mathrm{F}}(k_{01})=\int \mathrm{d}\mathbf{q}\,Q^{\mathrm{F}}(\tfrac{1}{2}\mathbf{K}+\mathbf{q}, \tfrac{1}{2}\mathbf{K}-\mathbf{q})g^2(q)/[k_{01}^2-q^2+i\eta] \tag{A.3}$$

occurring in the denominator in eq. (43). In fact, we have to replace $I_F(k_{01})$ by

$$I(k_{01}) = \int d\mathbf{q}[1-P(|\tfrac{1}{2}\mathbf{K}+\mathbf{q}|)][1-P(|\tfrac{1}{2}\mathbf{K}-\mathbf{q}|)]g^2(q)/[k_{01}^2-q^2+i\eta]. \quad \text{(A.4)}$$

The effect of the difference $I(k_{01})-I_F(k_{01})$ on $t_{01}$ is clearly very small at high energies, as the effect of the exclusion principle on $t_{01}$ in that limit is negligible. On the other hand, this effect is not negligible at small energies. In particular with the approximation $K \approx 0$ we find that the correction to Re $I_F$ is 1.1 per cent at $k_{01} = 1.5k_F$ and 37 per cent at $k_{01} = 1.1k_F$. This diminishes a little the real part of the optical potential. In the same approximation there is almost no correction to Im $I_F$ for $k_{01} > k_F$. For $k_{01} = 0.8k_F$ the correction is $-7.7$ per cent which corresponds to a small increase of the imaginary well depth. The latter result is consistent with that of Brueckner[7]).

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*Nuclear Physics* **13** (1959) 642—654; © *North-Holland Publishing Co., Amsterdam*
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# GAMMA RAY CASCADES IN $O^{15}$

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Received 4 August 1959

**Abstract:** The gamma rays from the reaction $N^{14}(p, \gamma)O^{15*}$ have been investigated over the proton energy range from 950 to 1560 keV. It is found that the 8.29-MeV level (1060-keV resonance) in $O^{15}$ cascades weakly through a level at 7.17 MeV and through the known level at 6.15 MeV, in addition to the transitions already known to the 5.2-MeV levels and the ground state. The radiative widths of the 8.29-MeV level for transitions to the ground state, 5.2-MeV levels, 6.15-MeV level, and 7.17-MeV level are respectively 0.54 eV, 0.33 eV, 0.06 eV and 0.02 eV. The non-resonant production of $O^{15}$ in the above proton energy range is almost entirely accounted for by a transition through the 6.79-MeV level in $O^{15}$. At the 8.75-MeV level (1550-keV resonance) the transitions to the ground state, 5.2-MeV states, 6.8-MeV states, and 7.17-MeV state all appear to be weakly resonant.

## 1. Introduction

A comparison [1]) of the excited states of $O^{15}$ and $N^{15}$ up to an energy of approximately 9 MeV leads to the conclusion that there is at least one state yet to be found in $O^{15}$ at an excitation close to 7 MeV. There is little doubt that the pair of levels at 5.27 and 5.31 MeV in $N^{15}$ correspond to the pair of levels [2]) at 5.25 and 5.19 MeV in $O^{15}$. The $N^{15}$ level at 6.33 MeV has a counterpart in the 6.15-MeV level of $O^{15}$, both being of negative parity, and expected on the shell model to be the $p_{\frac{1}{2}}$ hole in an otherwise complete s and p shell. The 8.32 and 8.57-MeV levels of $N^{15}$ have been correlated [3]) with the 7.56 and 8.29-MeV levels respectively of $O^{15}$ on the basis of reduced width comparisons and of the expected level shifts. The known spin and parity assignments are consistent with these correlations. This leaves three levels in $N^{15}$ at 7.16, 7.31 and 7.57 MeV to be associated with only two known levels in $O^{15}$ at 6.79 and 6.86 MeV. Of these, the 6.79-MeV level of $O^{15}$ has been correlated [4]) with the 7.31-MeV level of $N^{15}$, again on the basis of reduced widths and level shifts, together with compatibility of spin and parity assignments. It is not known which of the two remaining levels in $N^{15}$ (7.16 MeV and 7.57 MeV) is the partner of the 6.86-MeV level in $O^{15}$, but it is clear that the missing level in $O^{15}$ lies somewhere in the region around 7 MeV. The level shift due to the boundary conditions is such that the $O^{15}$ levels are depressed relative to their partners in $N^{15}$, those of high spin,

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†† Supported in part by the joint program of the Office of Naval Research and the U.S. Atomic Energy Commission.

in general, suffering little shift [3]). Therefore, we expect the missing level to be below 7.58 MeV. Both the 7.16-MeV and the 7.57-MeV levels of $N^{15}$ show an $l_n = 2$ stripping distribution in the $N^{14}$(d, p) reaction [5]). The spin and parity of the corresponding states in $O^{15}$ would be $\leqq \frac{7}{2}^+$.

No suitable particle reaction of sufficiently high $Q$-value to investigate narrow levels in the 7-MeV region of excitation of $O^{15}$ was available with the present facilities of the Kellogg Radiation Laboratory. The method chosen to look for the missing $O^{15}$ level was to populate higher states in $O^{15}$ by proton bombardment of $N^{14}$ and to investigate the gamma-ray decays of these states. Of the higher states available, a promising one is the 8.29-MeV state, which occurs as a resonance for protons of energy 1064 keV [6]). The resonance is strong, and the non-resonant yield may be made small in comparison with the resonant yield. The 8.29-MeV state is sufficiently low in energy that its gamma spectrum should not be too complex, and with the known $J = \frac{3}{2}^+$ of the 8.29-MeV level, there is a reasonable chance of observing transitions to any $J = \frac{1}{2}, \frac{3}{2}$ and $\frac{5}{2}$ states below it in energy.

The exact position of the missing level in $O^{15}$ is of considerable astrophysical interest because it is close to the proton threshold at 7.300 MeV. If it falls in the excitation range from 7.31 to 7.35 MeV, it will be able to affect the rate of the $N^{14}(p, \gamma)$ reaction in the carbon-nitrogen cycle. At present, this reaction is believed to set the rate of the C—N cycle, but a resonance in the region of stellar energies would transfer control of the rate of the cycle to the $C^{12}(p, \gamma)$ reaction. This would influence the effective operating temperature within a star and also affect the distribution of material among the various elements taking part in the C—N cycle.

The decay of $O^{15*}$ formed at the 1064-keV resonance in $N^{14}+p$ has been studied by Li [7]), who finds strong gamma rays of energy 8.34 MeV, a strong transition through a level at 5.2 MeV, and a weak transition through a level at 6.8 MeV. The angular distribution of the ground state radiation and the angular correlations of the cascade through the 5.2-MeV level (or levels) have been studied by Gorodetzky *et al.*[8]).

The non-resonant proton capture reaction $N^{14}+p$ has been studied mainly by the $O^{15}$ produced. Hagedorn *et al.*[9]) have suggested that the process responsible for the non-resonant cross section is direct capture. It was felt to be worthwhile to investigate the non-resonant radiation in more detail, both in the hope of learning more about the capture process itself and with the thought that the missing level in $O^{15}$ might be involved.

## 2. Method

The target used was of tantalum nitride, formed by heating 0.010-inch tantalum sheet to a bright red heat in an atmosphere of commercial ammonia. The target was mounted at right angles to the proton beam supplied by the

1.5-MV electrostatic accelerator in the Kellogg Radiation Laboratory. The target thickness was such that 1100-keV protons lost about 40 keV in passing through the $N^{14}$ layer. Oil vapor contamination of the target during bombardment was negligible, but it was necessary to limit the proton beam to 3 microamps in order to avoid losing nitrogen from the target.

The gamma-ray detection equipment consisted of two NaI(Tl) crystals, one $4\times4$ inches used for gamma-ray energies above 4 MeV, and the other $1.5\times1.5$ inches used for the lower energy gamma rays. These crystals were used in coincidence, with a single-channel analyzer selecting pulses (from the 4-inch crystal) in a certain energy range, and using these pulses to gate a 100-channel pulse height analyzer. The 100-channel analyzer displays the output of the 1.5-inch crystal that is in time coincidence with pulses from the 4-inch crystal satisfying the requirements of the single-channel analyzer.

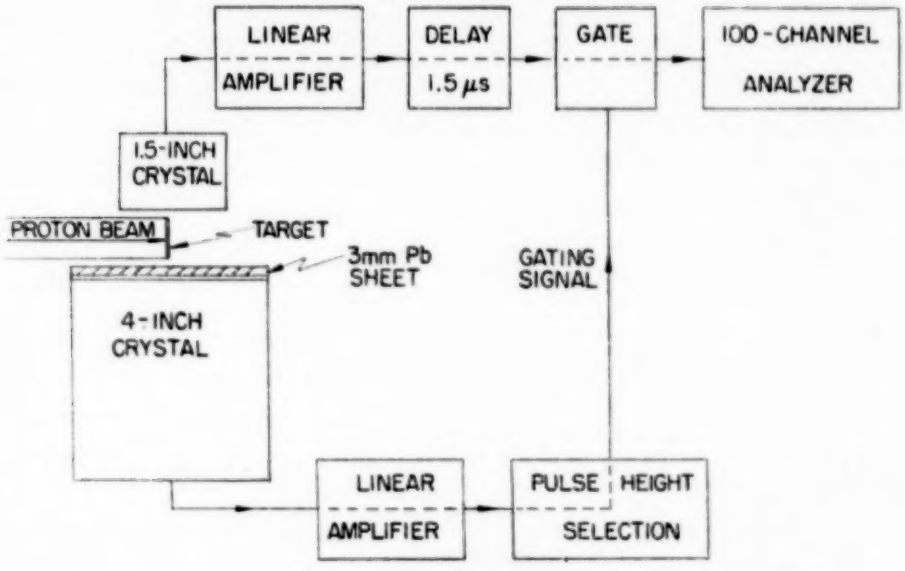


Fig. 1. The target-detector arrangement and the electronic block diagram.

The two sodium-iodide crystals were arranged 180° apart and each at 90° to the proton beam direction. The solid angle subtended by each counter was made as large as possible by moving the crystals close together with the target spot between them on the axis of both crystals. Under such conditions, the low energy end of any coincidence spectrum is distorted by true coincidences in which a gamma ray converted in one crystal produces a beta particle that is scattered out of the first crystal into the second one. To reduce the importance of this effect, a sheet of lead, 3 mm thick, was interposed between the crystals. The arrangement used is shown in fig. 1. The high energy gamma rays to the 4-inch crystal pass through this lead sheet, but the improvement in the low energy coincidence spectrum is very noticeable.

By altering the settings of the single-channel analyzer, the various low energy gamma rays may be displayed relatively free from confusing higher energy gamma rays. The energy calibration of both crystals was made with radioactive sources, supplemented by the 4.43-MeV and 8.29-MeV gamma rays produced in the natural nitrogen target during the proton bombardment.

The coincidence spectra were investigated at several different proton energies from 1045 keV to 1560 keV with special attention to the 1064-keV resonance. In addition, the high energy spectrum in the 4-inch crystal was displayed directly on the 100-channel analyzer at intervals of 100 keV in proton energy in the above energy range. Because of the 0.38 per cent $N^{15}$ in the natural nitrogen target, the 4.43-MeV gamma ray following $N^{15}(p, \alpha_1)C^{12*}$ dominated the low energy part of the singles spectrum over the whole energy range.

## 3. Results

The proton energy was initially set to 1075 keV, so that large numbers of 8.29-MeV gamma rays as well as others of lower energies were entering both crystals. With the single-channel analyzer set to accept pulses in the energy range from 7.7 to 8.6 MeV, the low energy coincidence spectrum was observed to have only the expected single peak due to annihilation quanta. The single-channel analyzer setting was high enough that no real coincidences involving successive nuclear radiations from $O^{15*}$ should have been observed. On the other hand, the single-channel count rate was equal to or greater than the rates observed in later coincidence runs, so that the accidental coincidence rate observed with the single-channel analyzer setting 7.7—8.6 MeV will be an upper limit. The events recorded with this single-channel analyzer setting are pair production by an 8.29-MeV gamma ray in the 4-inch crystal of most of the negaton and positon energy. However, one of the quanta from the positon annihilation escapes from the 4-inch crystal but is captured in the 1.5-inch crystal. A number of back-scattered quanta from Compton events in the 4-inch crystal (originated by 8.29-MeV gamma rays) are also recorded as coincidences. The accidental coincidence rate plus the background rate (above an energy of 0.6 MeV in the 1.5-inch crystal) per channel, as displayed on the 100-channel analyzer, is less than 0.3 per cent of the count rate at the 0.51-MeV peak.

For the proton energy of 1075 keV, the single-channel analyzer was set sucessively at the energy ranges 4.2—5.6 MeV, 5.4—6.2 MeV, 6.4—7.0 MeV, and 6.4—7.6 MeV. Because the single-channel analyzer actually performed a pulse height selection, and because of the spread in pulse heights even of a mono-energetic gamma ray, the coincidence spectra for the lower energy ranges of the single-channel analyzer contain all the gamma rays revealed by the higher settings, and in addition, sometimes allow a leak-through of pulses due to a gamma ray of energy slightly below the nominal lower limit of the

single-channel analyzer. Therefore, the coincidence spectrum interpretation is made easier by inspecting the results of the single channel analyzer settings in order of decreasing energy.

For the settings 6.4—7.0 MeV and 6.4—7.6 MeV, the shapes of the low energy coincidence spectra were essentially identical. Fig. 2, for the setting 6.4—7.6 MeV, shows the annihilation peak, a gamma ray of $1.50\pm0.02$ MeV, and indications of a gamma ray with an energy close to 1.1 MeV. The 1.50-MeV gamma ray of fig. 2 may be identified with the transition from 8.29 MeV to the 6.79-MeV level in $O^{15}$. This transition was first observed by $Li^7$ who reported the energy as $1.46\pm0.03$ MeV. The present energy measurement is sufficiently

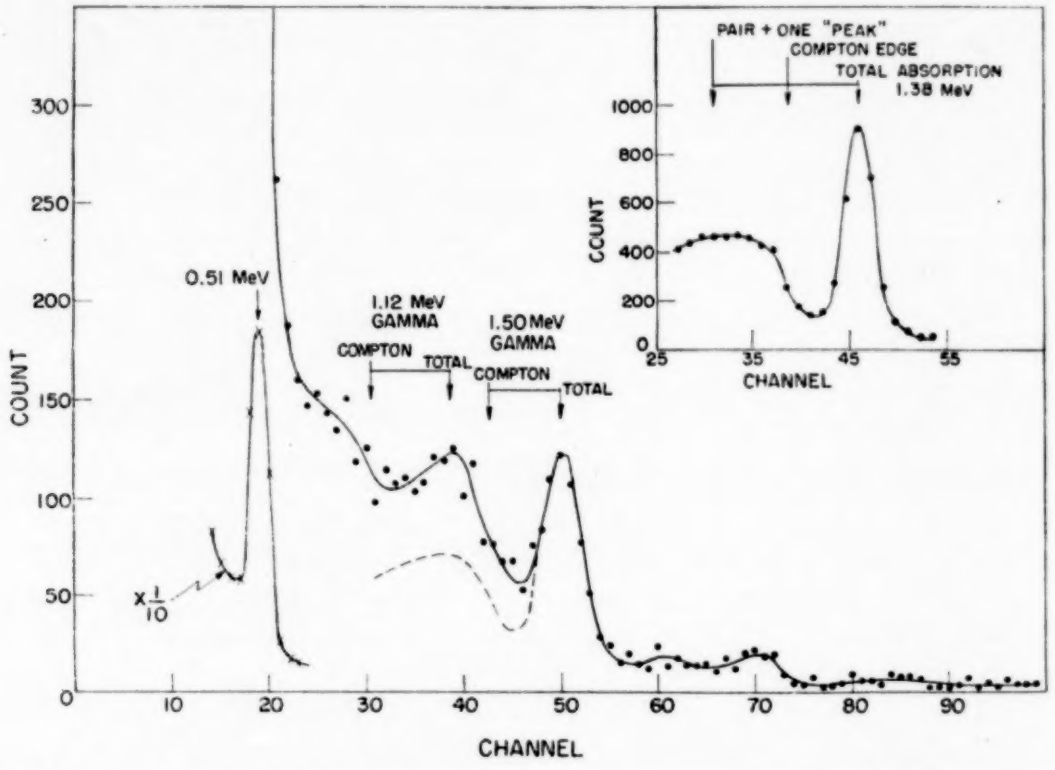


Fig. 2. A spectrum from the 1.5-inch crystal of low energy gamma rays from $N^{14}+p$ in coincidence with pulses in the energy range 6.4—7.6 MeV. The proton energy is 1075 keV. The dashed curve is the expected spectrum shape of a 1.50-MeV gamma ray. The excess pulses below the Compton edge of the 1.50-MeV gamma ray are attributed to a 1.12-MeV gamma ray. The annihilation quanta do not contribute a significant number of pulses in this energy region, nor can the excess pulses be accounted for by accidental coincidences. It is possible that the transition 8.29—6.86 MeV is weakly resonant, but the 1.43-MeV gamma ray from this transition also cannot account for the extra pulses. The small peaks at channels 60 and 70 are caused by a 2.10-MeV gamma ray shown more clearly in fig. 3. The inset in this figure shows a 1.38-MeV gamma ray (coincidence) spectrum observed with the same 1.5-inch crystal at the same target-detector distance. The 1.38-MeV gamma ray is produced in the $N^{14}(p,\gamma)$ reaction at the 277-keV resonance.

precise to exclude the possibility that a large proportion of the transition is to the 6.86-MeV level. The 1.1-MeV gamma ray, for which the best energy measurement is $1.12\pm0.05$ MeV, is thought to be a transition from the 8.29-MeV level

to a new level located at 7.17 MeV in $O^{15}$. The fact that the 1.1-MeV gamma ray is resonant at or near 1064-keV proton energy and that the low energy gamma-ray spectra at the 1064-keV resonance consistently show this gamma ray (when not obscured by higher energy gamma rays) supports the assignment of this gamma ray to a transition in $O^{15}$. About fifty targets were used over a period of four months to obtain the spectra from which this conclusion is drawn. Attempts to identify the 1.1-MeV gamma ray (which must be in coincidence with a gamma ray of energy between 6.4 and 7.2 MeV) with a transition in a contaminant have been made, but no such contaminant has been identified. The singles spectra over the energy range from 1020 keV to 1560-keV proton

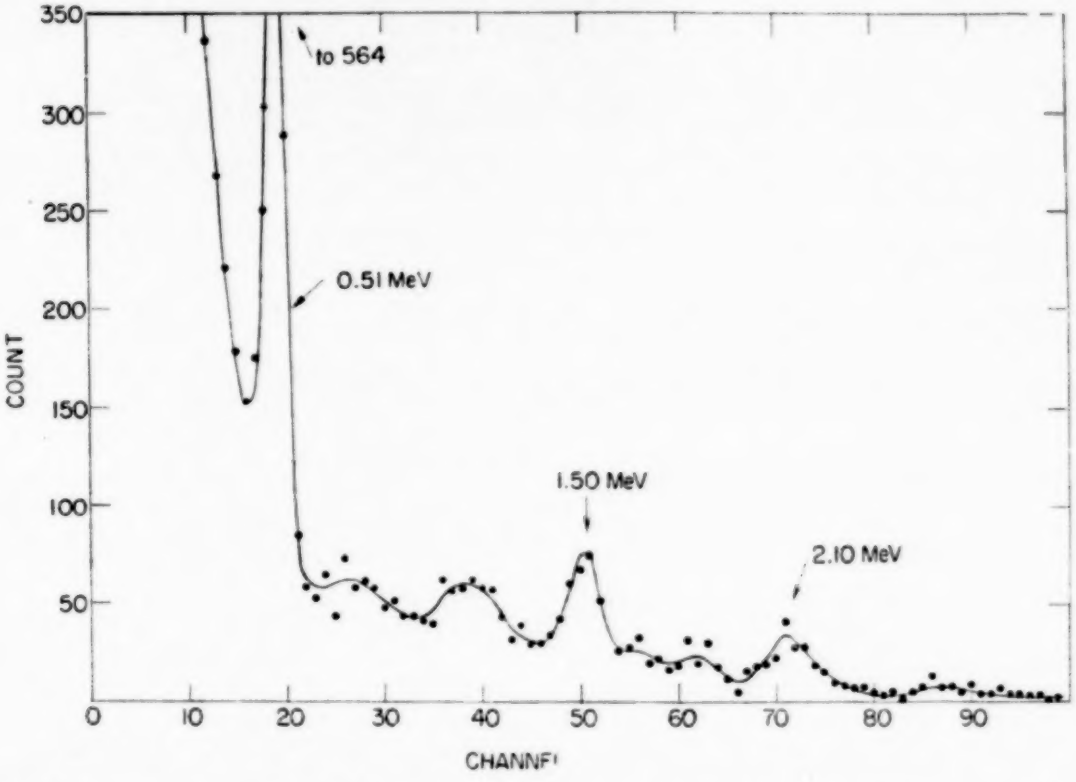


Fig. 3. A spectrum from the 1.5-inch crystal of low energy gamma rays from $N^{14}+p$ in coincidence with pulses in the energy range 5.4—6.2 MeV. The proton energy is 1075 keV.

energy show no signs of appreciable contamination.

Fig. 3, for the single channel analyzer setting of 4.5—6.2 MeV, shows a 2.10-MeV gamma ray in addition to the previously mentioned gamma rays of fig. 2. The 2.10-MeV gamma ray is resonant at 1064 keV, and is to be associated with a transition in $O^{15}$ from the 8.29-MeV level to the 6.15-MeV level. This transition has not previously been observed as it is only a small fraction of the total gamma-ray yield.

The transition from the 8.29-MeV level to the levels at 5.2 MeV is well known to be strongly resonant. This transition was investigated with the single-

channel analyzer setting 4.2—5.6 MeV to determine its strength relative to the other resonant transitions, but the gamma-ray energy measurement of $3.05\pm 0.05$ MeV was not sufficiently accurate to distinguish which of the pair of levels at 5.19 and 5.25 MeV is involved. Since the 8.29-MeV level is $\frac{3}{2}^+$ and the 5.19 and 5.25-MeV levels are probably $\frac{1}{2}^+$ and $\frac{5}{2}^+$ respectively, both levels may be involved in the cascade.

The relative intensities of the gamma rays resonant at 1064 keV proton energy were determined first by measuring a coincidence spectrum showing the relative proportions of 1.1 and 2.1-MeV radiation, correcting approximately for the different efficiencies of detection of the high energy gamma rays caused by the single-channel analyzer setting, but ignoring the unknown angular correlation effects which are considerably reduced because the NaI crystals are so close to the target. Then the sum of the low energy radiations was compared with the 3.05-MeV radiation, also in a coincidence spectrum. Finally, the ratio of 3.05-MeV radiation to 8.29-MeV radiation was measured in a singles spectrum that was the difference of two spectra at and below the 1064-keV resonance. The difference was taken to reduce the amount of 4.43-MeV radiation from the $N^{15}(p, \alpha, \gamma)$ reaction, which has a broad resonance in the same energy region. Table 1 shows the relative intensities of the various gamma rays from the 8.29-MeV level of $O^{15}$.

TABLE 1

Radiative widths of the 8.29-MeV level in $O^{15}$

| $\gamma$ Transition from 8.29-MeV level to | Ground State | 5.2-MeV State(s) | 6.15-MeV State | 7.17-MeV State | Total Radiative Width |
|---|---|---|---|---|---|
| Resonance partial widths (eV) | 0.54 | 0.33 | 0.06 | 0.02 | 0.95 [a] |

[a] Ref. [6]). The later correction of this figure in ref. [9]) is in error because the workers of ref. [6]) obtained the thick target yield of $O^{15}$, and this, in the present case, is independent of the actual resonance width.

The non-resonant radiation, with a yield of approximately 6 per cent of the resonant reaction yield for a target thickness of 40 keV, is shown in figs. 4 and 5 measured at an energy of 1045 keV. Fig. 4 shows a gamma ray of energy $1.47\pm 0.02$ MeV, with no sign of the 1.1-MeV gamma ray observed at the 1064-keV resonance. The expected spectrum shape of a 1.47-MeV gamma ray agrees well with that shown in fig. 4. The change in energy from 1.50 to 1.47 MeV in the non-resonant radiation is consistent with the change in proton bombarding energy from 1075 to 1045 keV. The 1.5-MeV radiation is in coincidence with radiation of energy $6.8\pm 0.1$ MeV, in agreement with the assignment of both gamma-rays to $O^{15}$. The intensity of the 6.8-MeV gamma ray, for which a

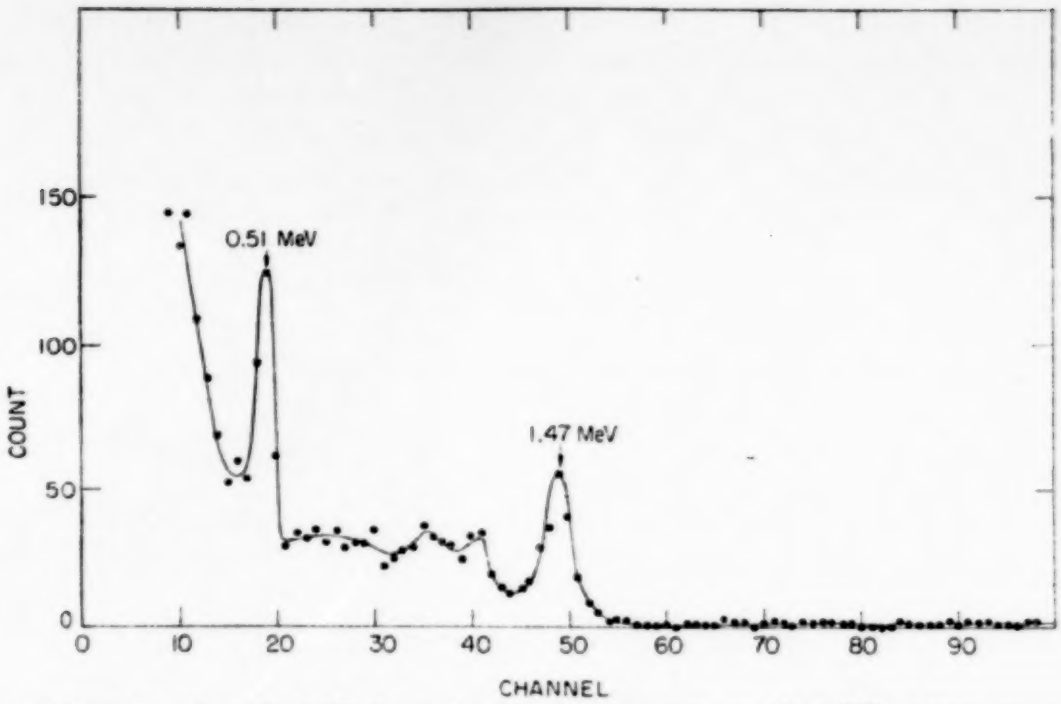


**Fig. 4.** A spectrum from the 1.5-inch crystal of low energy gamma rays from $N^{14}+p$ (non-resonant) in coincidence with high energy pulses in the energy range 6.4—7.0 MeV. The proton energy is 1045 keV. A comparison with the inset of fig. 2 shows that the spectrum displayed here is that of a pure 1.47-MeV gamma ray, with no extra pulses in the region of 1.1 MeV.

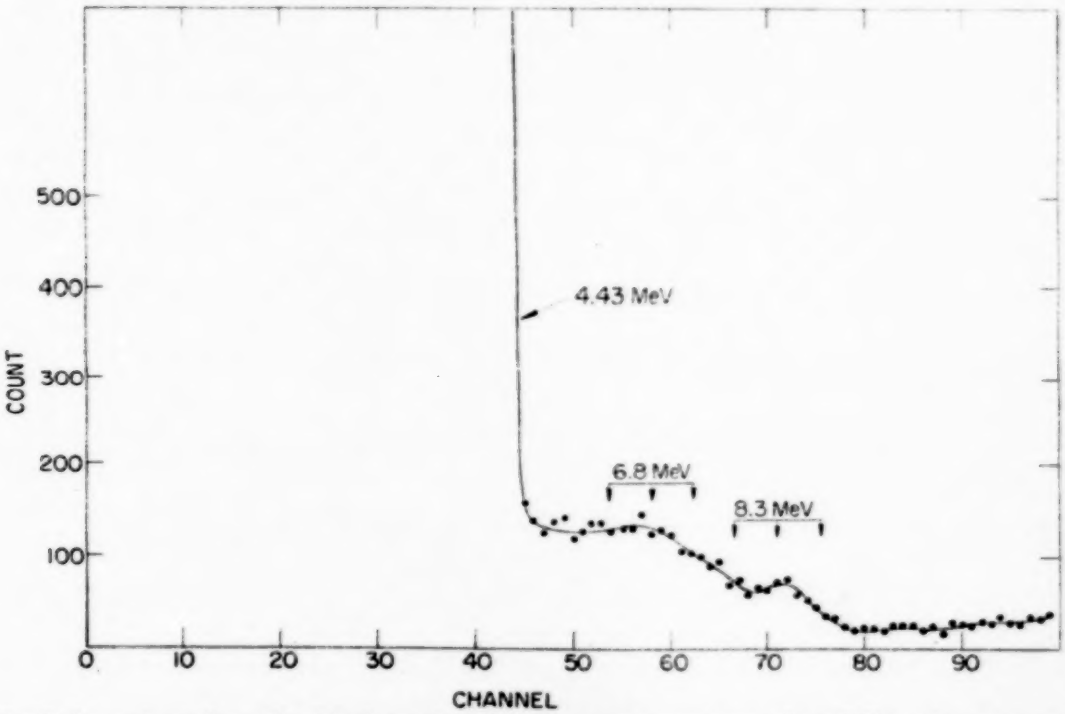


**Fig. 5.** A spectrum from the 4-inch crystal of high energy gamma rays from $N^{14}+p$ (non-resonant) at a proton energy of 1045 keV. The 4.43-MeV radiation in this singles spectrum is from the $N^{15}(p, \alpha\gamma)$ reaction in the natural $N^{15}$ content of the target.

singles spectrum from the 4-inch crystal is shown in fig. 5, corresponds to a non-resonant cross section at 1045 keV of approximately 4 $\mu$b. The non-resonant yield of $O^{15}$ measured by Duncan and Perry [6]) also corresponds to a cross section of approximately 4 $\mu$b. This agreement shows that the cascade through the 6.79-MeV level is responsible for the major fraction of the non-resonant production of $O^{15}$. The yield of 8.3-MeV radiation shown in fig. 5 is largely yield from the tail of the 1064-keV resonance. The non-resonant cross section for ground state radiation is less than or equal to 1 $\mu$b. An upper limit

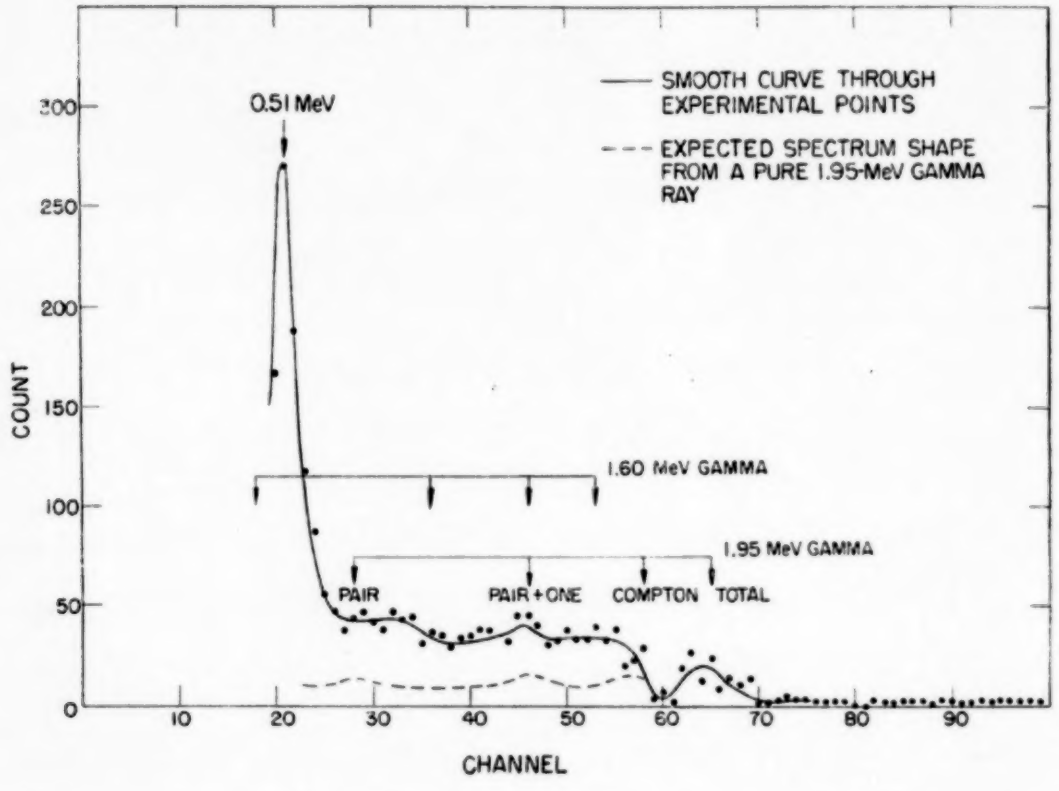


Fig. 6. A spectrum from the 1.5-inch crystal of low energy gamma rays from $N^{14}+p$ in coincidence with pulses in the energy range 6.4—7.0 MeV. The proton energy is 1560 keV.

of 1 $\mu$b applies to cascades through the 6.15-MeV level. The latter limit is obtained from low energy coincidence spectra taken below the resonance as well as from fig. 5.

As the proton energy is raised, the cascade through the 6.8-MeV level continues to dominate the non-resonant cross section. There is however, a small amount of 5.3-MeV radiation and ground state radiation present. At the 1550-keV resonance [1]), there is an increase in the yield of all three high energy radiations, all apparently being resonant. The low energy coincidence spectrum at 1560 keV (with the single-channel analyzer set at 6.4—7.0 MeV) shows, in fig. 6, a gamma ray of energy $1.95\pm0.05$ MeV, consistent in energy with the

transition to the 6.79-MeV level in $O^{15}$. Below the Compton edge of this 1.95-MeV gamma spectrum, there are many more pulses than are to be expected from a single 1.95-MeV gamma ray. The spectrum is similar to that shown in fig. 2, but the excess of pulses just below the Compton edge is more marked in the spectrum taken at 1560 keV. The latter spectrum is consistent with a gamma-ray energy of $1.60 \pm 0.05$ MeV. This gives agreement with the previous estimate of 7.17 MeV for the position of the missing $O^{15}$ level.

An attempt was made to find the method of de-excitation of the 7.17-MeV

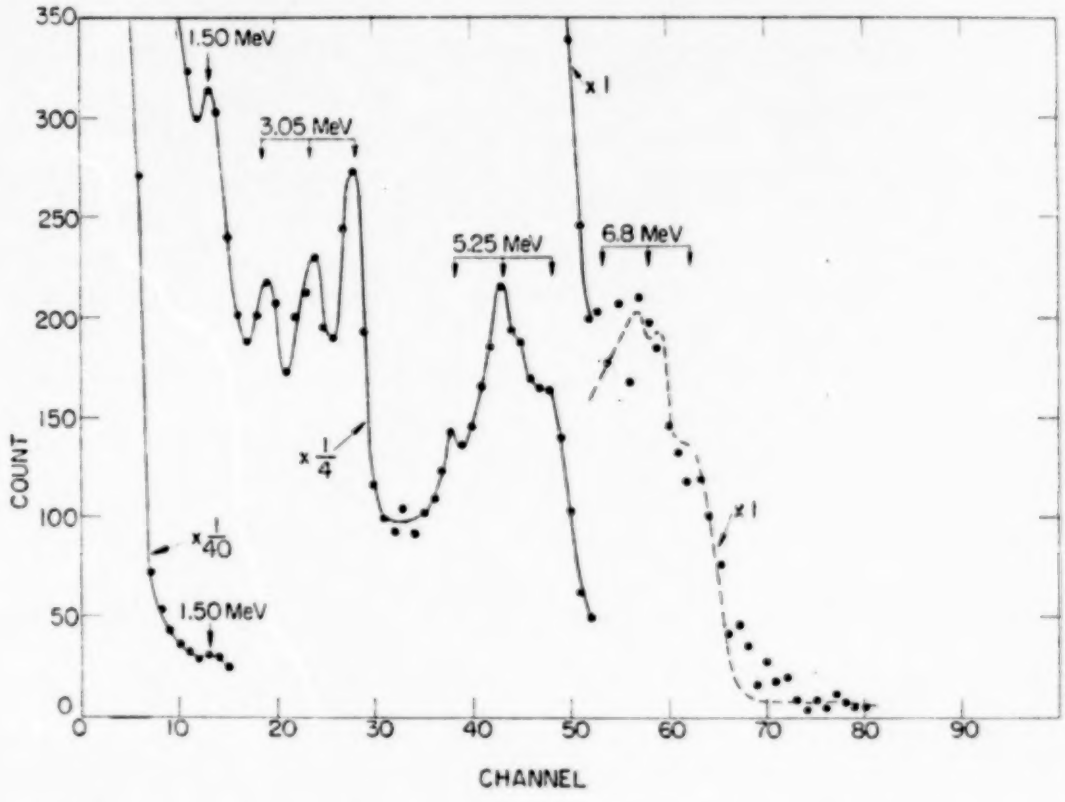


Fig. 7. A spectrum from the 4-inch crystal of gamma rays in coincidence with pulses between 0.8 and 1.2 MeV in the 1.5-inch crystal. The dashed curve represents the expected shape of a pure 6.8-MeV gamma ray spectrum.

level. The high energy spectrum in coincidence with pulses from 0.80 to 1.20-MeV energy was measured at the 1064-keV resonance. The result is shown in fig. 7, on which is also marked the expected shape of a 6.8-MeV gamma ray. There are some pulses above the limit expected for a 6.8-MeV gamma ray. From fig. 7, the intensity of any 7.2-MeV gamma ray must be less than 25 per cent of the intensity of the 6.8-MeV gamma ray. The expected intensity ratio is approximately 20 per cent. It may be possible with a thinner target to improve this expected intensity ratio, and thus obtain more definite information on the decay of the 7.17-MeV level.

## 4. Discussion

It has been suggested that the non-resonant radiation, now known to be a cascade through the 6.79-MeV level, is the result of a direct capture process [9]). Under certain conditions, this direct capture cross section may be calculated with fair accuracy [10]), and the $N^{14}$+p reaction appears to satisfy the required conditions. The 6.79-MeV level is thought to have a large proton reduced width because it occurs as a prominent threshold in the $N^{14}$(d, n) reaction [4]). It is known to have a spin [2]) of $\frac{3}{2}$ with positive parity [11]), and the shell model structure expected would be a mixture of 2s and 1d proton orbits with an $N^{14}$ core mostly in the ground state. The direct capture process would not involve that part of the wave function corresponding to a core of $N^{14}$ in an excited state. The direct capture cross section for an incoming p-wave proton captured to an s orbit with emission of E1 radiation has been computed, using a formula suggested by Christy and Duck [10]), for a proton energy of 1000 keV. The computed cross section of 6 $\mu$b is an approximate upper limit which may be in error by up to a factor of two. It assumes a pure s-wave proton orbit and an $N^{14}$ ground state core for the 6.79-MeV state. The type of transition calculated should account for the most important transition of the direct capture process. The agreement obtained between the calculated direct capture cross section and the experimental value for the non-resonant cross section is very good, in accord with similar results obtained by other workers [12, 13]) for the $O^{16}$(p, $\gamma$) and $Ne^{20}$(p, $\gamma$) reactions.

Further information about the 6.79-MeV state in $O^{15}$ may be available from the angular distribution of the direct capture radiation to the 6.79-MeV level. This angular distribution may give the relative proton reduced widths for s and d orbits in the 6.79-MeV level. The following transition to the ground state will contain in its angular distribution the channel spin mixture of the d-wave component of the wave function for the 6.79-MeV level. Results of this nature would permit a comparison of the shell model calculations of Halbert and French [14]) or Lane's simple model [15]) with the experimental information in an energy region where the association of shell model levels with the observed levels is at present not completely satisfactory. Experimentally, the resolving of the 1.5-MeV gamma ray from a singles spectrum will be difficult, but may not prove impossible with nitrogen depleted in $N^{15}$ by a factor of 30 and a choice of energy so as to avoid the prominent short range alpha-particle resonances in $N^{15}$+p.

The identification of a level at 7.17 MeV in $O^{15}$ leads to no important changes in the astrophysical calculations of the rate of the carbon-nitrogen cycle. The rate of energy generation is still controlled by the $N^{14}$(p, $\gamma$) reaction. The capture at stellar energies appears to be almost entirely due to the non-resonant process. Calculations have been made by Morton [16]) on the effect on the main

sequence of transfer of control of the C—N cycle from $N^{14}(p, \gamma)$ to $C^{12}(p, \gamma)$. The results would indicate a sideways shift of the main sequence on the mass-luminosity diagram. The shift would be of the order of the experimental spread in width of the main sequence, and the absence of such a shift is evidence for the absence of a level in the critical region of $O^{15}$ just above 7.300 MeV.

## 5. Conclusion

The observed transitions in $O^{15}$ which are the subject of this report are displayed in fig. 8. The correlations of $O^{15}$ and $N^{15}$ levels that are reasonably

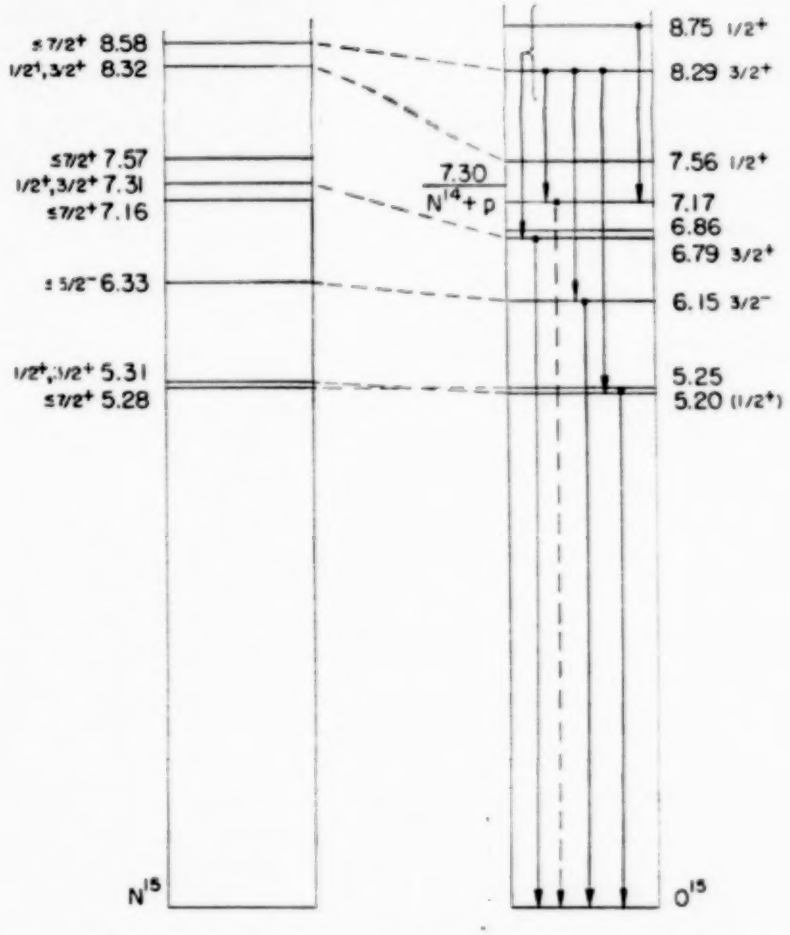


Fig. 8. An energy level diagram of $N^{15}$ and $O^{15}$, showing levels below 9 MeV and the transitions in $O^{15}$ assigned to the gamma rays observed. The correlations between levels in $N^{15}$ and $O^{15}$ that are reasonably well established are indicated by dashed lines.

certain are shown in the same figure, and it would appear that there are the same number of levels to be associated in both nuclei. However, there are conflicting reports of the mode of de-excitation of the 7.165-MeV level in $N^{15}$ produced on the one hand by $N^{14}(d, p)$ [17]) or $N^{14}(n, \gamma)$ [18]) and on the other hand by $C^{14}(p, \gamma)$ [19]). A possible explanation is the presence in $N^{15}$ of two levels, as

yet unresolved, in the region of 7.16-MeV excitation. This would mean yet another level (besides the new level at 7.17 MeV) to be found in $O^{15}$. The present results cannot say definitely the number of levels in $O^{15}$, because the extra gamma rays (in excess of those already accounted for by known levels) are not sufficiently well resolved.

The authors wish to thank Professor R. F. Christy, Professor W. A. Fowler, and Professor T. Lauritsen for much helpful advice and assistance. The comments on the manuscript by Professor S. Bashkin were much appreciated.

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# SUPERFLUIDITY AND THE MOMENTS OF INERTIA OF NUCLEI

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Received 11 April 1959

**Abstract:** A method is presented which permits one to study superfluidity in finite size systems. Moments of inertia are computed by this method in the quasi-classical approximation and satisfactory agreement with the observed values is obtained. The calculated increase of the moment of inertia upon transition from even to odd-mass nuclei and also the gyromagnetic ratio for rotating nuclei are in agreement with the experiments. These results thus confirm the assumption of superfluidity of nuclear matter.

## 1. Introduction

Depending on the nature of the excited states, systems consisting of interacting Fermi particles can be divided into two classes. If repulsive forces are dominant the excitations in the system will be the same as those for free Fermi particles, but the effective mass will depend on the interaction forces between the particles. In the case of attractive forces "correlated pairs" are formed and these lead to the appearance of an energy gap and to superfluidity [1–3]). It may be expected that this second type of single particle excitation exists in nuclei. As an experimental confirmation one may mention, in particular, the fact that in even-nuclei the energy of the first single-particle excitation is several times larger than it would be for systems of the first type and is also greater than for odd-mass nuclei. The "pairing" energy, as determined from mass defect data, also speaks in favour of this phenomenon.

However, the existence of correlated pairs and superfluidity are especially manifested in the moments of inertia of nuclei. The latter are 2—3 times smaller than those computed by formulas for the moments of inertia of rigid bodies and this is the most direct proof that nuclear matter is in a superfluid state. It would therefore appear to be of especial interest to calculate the moments of inertia of nuclei on the basis of the present theory of superfluidity of Fermi systems. The formalism of this theory has been developed for uniform infinite systems [4]), whereas in nuclei the finite dimensions of the system are very important.

A method is presented below which permits one to study superfluidity in

finite size systems. Moments of inertia are computed by this method in the quasi-classical approximation and satisfactory agreement with the observed values is obtained.

The calculated increase of the moment of inertia upon transition from even to odd-mass nuclei and also the gyromagnetic ratio for rotating nuclei are in agreement with the experiments. These results thus confirm the assumption of superfluidity of nuclear matter.

It should be noted that superfluidity of nuclear matter may lead to some interesting cosmological phenomena if stars exist which have neutron cores. A star of this type would be in a superfluid state with a transition temperature corresponding to 1 MeV.

## 2. Method of Treatment of Finite Size Systems

### 2.1. FUNDAMENTAL EQUATIONS

The method proposed by L. P. Gorkov can conveniently be applied to study the superfluidity of a finite size system.

Along with the usual Green's function for a single particle

$$G(1, 2) = -i(\phi_N{}^0 \mathrm{T}\psi(1)\psi^+(2)\phi_N{}^0) \tag{1}$$

a function $F(1, 2)$ defined by the formula

$$F(1, 2) = (\phi_N{}^0 \mathrm{T}\psi^+(1)\psi^+(2)\phi^0_{N-2}) \exp(-2i\varepsilon_0 t) \tag{2}$$

is introduced. Here $\phi_N{}^0$ and $\phi^0_{N-2}$ are the ground state eigenfunctions of a system consisting of $N$ and $N-2$ interacting particles, T is a time-ordering operator, $\psi(\mathbf{r}, t)$, $\psi^+(\mathbf{r}, t)$ are, respectively, particle annihilation and creation operators in the Heisenberg representation.

A set of equations relating $G$ and $F$ was obtained by Gorkov for a weak, $\delta$-like interaction between the particles. A similar set of equations for arbitrary interaction between the particles can be obtained by field theory methods.

Another way of obtaining a set of equations for arbitrary interaction is to apply Gorkov's method to a system of interacting quasiparticles. The following set of equations for $F$ and $G$ is obtained†:

$$\begin{aligned}
&[\varepsilon - H(\mathbf{r})]G(\mathbf{r}\mathbf{r}'\varepsilon) = \delta(\mathbf{r}-\mathbf{r}') + i\Delta(\mathbf{r})F(\mathbf{r}\mathbf{r}'\varepsilon),\\
&[\varepsilon + H^*(\mathbf{r}) - 2\varepsilon_0]F(\mathbf{r}\mathbf{r}'\varepsilon) = -i\Delta^*(\mathbf{r})G(\mathbf{r}\mathbf{r}'\varepsilon);\\
&\Delta(\mathbf{r}) = \gamma(\mathbf{r})\int F^*(\mathbf{r}\mathbf{r}\varepsilon)\frac{\mathrm{d}\varepsilon}{2\pi}.
\end{aligned} \tag{3}$$

Here, the functions $G(\mathbf{r}\mathbf{r}'\varepsilon)$ and $F(\mathbf{r}\mathbf{r}'\varepsilon)$ are the Fourier transforms, with respect to the last variable, of the functions $G(\mathbf{r}\mathbf{r}', t-t')$ and $F(\mathbf{r}\mathbf{r}', t-t')$, defined by formulas (1) and (2). The quantity $\varepsilon_0$ is given by the expression

† The sign * means complex conjugation.

$\varepsilon_0 = \frac{1}{2}(E_N^0 - E_{N-2}^0)$, where $E_N^0$ is the energy of the ground state of a system of $N$ particles. As thus defined, $\varepsilon_0$ differs from the chemical potential $\varepsilon'_0 = E_{N+1}^0 - E_N^0$ by the "pairing" energy. The quantity $\gamma(\mathbf{r})$ is defined through a function describing the interaction between quasiparticles.

The boundary conditions for the functions $G$ and $F$ with respect to $\mathbf{r}$ and $\mathbf{r}'$ are specified by the boundary conditions for the eigenfunctions of the operator $H = \hat{p}^2/2M_{\text{eff}} + U(\mathbf{r})$.

### 2.2. DENSITY MATRIX AND MEAN VALUES

Let us assume that a solution of set (3) has been found which satisfies the boundary conditions. Knowing $G(\mathbf{r}\mathbf{r}'\varepsilon)$ one can easily find the single particle density matrix

$$\rho(\mathbf{r}\mathbf{r}') = (\phi_N^0 \psi^+(\mathbf{r})\psi(\mathbf{r}')\phi_N^0). \tag{4}$$

This quantity can be employed to determine the mean value over the ground state of any quantity $A$ which is a sum over the particles $A = \sum_j A(\mathbf{r}_j \mathbf{p}_j)$. The mean value of $A$ is

$$\langle A \rangle = (\phi_N^0 \int \psi^+ A(\mathbf{r}\mathbf{p})\psi \mathrm{d}\mathbf{r}\, \phi_N^0) = \mathrm{tr}\, A\rho.$$

Comparing (4) with the definition of the function $G$ we get

$$\rho(\mathbf{r}\mathbf{r}') = -iG(\mathbf{r}\mathbf{r}'\tau)|_{\tau \to -0} = \int_c G(\mathbf{r}\mathbf{r}'\varepsilon)\frac{\mathrm{d}\varepsilon}{2\pi i} \tag{5}$$

where the contour c consists of the real axis and of an infinite semi-circle in the upper half plane.

### 2.3. THE FUNCTION $\gamma(\mathbf{r})$

The last of eqs. (3) contains a function $\gamma(\mathbf{r})$ which can be determined only on the basis of a detailed knowledge of the interactions between quasiparticles. The effective interaction between quasiparticles can be calculated only in the form of a series in some small parameter (e.g. a perturbation theory series if the particle interaction is small). For nuclei no such small parameter exists which would permit one to employ approximate methods. Hence calculation of $\gamma(\mathbf{r})$ in the nucleus is not possible by analytical methods.

### 2.4. SOLUTION FOR CONSTANT $\Delta$

The set (3) can be greatly simplified if the self-consistent potential entering $H$ is independent of the particle coordinates, that is, in the case of a rectangular well. The particle density is then constant throughout the volume and the function $\gamma(\mathbf{r})$ which depends on the coordinates only through the particle density is also constant. It can be verified that $F(\mathbf{r}\mathbf{r}\varepsilon)$ is also independent of $\mathbf{r}$ and therefore $\Delta(\mathbf{r})$ is constant throughout the volume of the system.

The solutions of eqs. (3) for a constant $\Delta$ can easily be obtained if the functions $G$ and $F$ are expanded in the eigenfunctions of the Hamiltonian $H$,

the boundary conditions for $G$ and $F$ thus being satisfied:

$$G(\mathbf{r}\mathbf{r}'\varepsilon) = \sum_{\lambda,\lambda'} G_{\lambda\lambda'}(\varepsilon)\varphi_\lambda^*(\mathbf{r})\varphi_{\lambda'}(\mathbf{r}'),$$
$$F(\mathbf{r}\mathbf{r}'\varepsilon) = \sum_{\lambda,\lambda'} F_{\lambda\lambda'}(\varepsilon)\varphi_\lambda^*(\mathbf{r})\varphi_{\lambda'}(\mathbf{r}'), \tag{6}$$
$$H\varphi_\lambda = \varepsilon_\lambda\varphi_\lambda.$$

Denoting

$$\varepsilon-\varepsilon_0 \to \varepsilon, \qquad H-\varepsilon_0 \to H, \qquad \varepsilon_\lambda-\varepsilon_0 \to \varepsilon_\lambda \tag{7}$$

we obtain from (3)

$$(\varepsilon-H)G = \delta(\mathbf{r}-\mathbf{r}')+i\Delta F,$$
$$(\varepsilon+H^*)F = -i\Delta^* G, \tag{8}$$
$$\Delta = \gamma\int_c F^*\frac{d\varepsilon}{2\pi}.$$

Substitution of (6) in (8) for a constant $\Delta$ yields †

$$G_{\lambda\lambda'} = \frac{\varepsilon+\varepsilon_\lambda}{\varepsilon^2-\varepsilon_\lambda^2-\Delta^2}\delta_{\lambda\lambda'}, \qquad F_{\lambda\lambda'} = -\frac{i\Delta}{\varepsilon^2-\varepsilon_\lambda^2-\Delta^2}\delta_{\lambda\lambda'}. \tag{9}$$

The poles of $G_{\lambda\lambda'}$ determine the energy levels of the single-particle excitations

$$\varepsilon = E_\lambda = \sqrt{\Delta^2+\varepsilon_\lambda^2}. \tag{10}$$

In order to find the density matrix the singularities of $G$ in the complex plane $\varepsilon$ must be known. In the denominator of $G$ one should replace $\varepsilon_\lambda$ by

$$\varepsilon_\lambda - i\delta\varepsilon_\lambda/|\varepsilon_\lambda|, \quad \delta \to +0$$

this being equivalent to the introduction of a vanishingly small damping of the states describing the particle and hole. Then $E_\lambda = \sqrt{\Delta^2+\varepsilon_\lambda^2} \to \sqrt{\Delta^2+\varepsilon_\lambda^2-2i\delta\varepsilon_\lambda^2/|\varepsilon_\lambda|} = E_\lambda - i\delta_1$. Thus, in the expression $\varepsilon^2-E_\lambda^2 = (\varepsilon-E_\lambda+i\delta_1)(\varepsilon+E_\lambda-i\delta_1)$ the first factor vanishes in the lower half-plane and the second in the upper half-plane of $\varepsilon$.

According to (5) the density matrix is

$$\rho_{\lambda\lambda'} = \int G_{\lambda\lambda'}(\varepsilon)\frac{d\varepsilon}{2\pi} = \frac{E_\lambda-\varepsilon_\lambda}{2E_\lambda}\delta_{\lambda\lambda'} = \nu_\lambda\delta_{\lambda\lambda'}. \tag{11}$$

We similarly find

$$\int_c F_{\lambda\lambda'}(\varepsilon)\frac{d\varepsilon}{2\pi} = -\frac{\Delta}{2E_\lambda}\delta_{\lambda\lambda'}.$$

Inserting this result in the last of eqs. (8) we find the relation between $\gamma$ and $\Delta$

$$1 = -\gamma\sum_\lambda \frac{1}{2E_\lambda}\varphi_\lambda^*(\mathbf{r})\varphi_\lambda(\mathbf{r}). \tag{12}$$

† The constant phase shift $\Delta$ in eq. (8) can be chosen arbitrarily and hence for $\Delta = \text{const}$ one may put $\operatorname{Im}\Delta = 0$.

## 2.5. WEAK PERTURBATION

We shall now solve equation set (8) for a weak perturbation $H \to H+V$. For first order terms in $V$ the equations are

$$\begin{aligned}(\varepsilon-H)G'-VG &= i\Delta' F+i\Delta F',\\ (\varepsilon+H)F'+V^*F &= -i\Delta'^*G-i\Delta G',\\ \Delta'^*(\mathbf{r}) &= \gamma\int F'(\mathbf{r}\mathbf{r}\varepsilon)\frac{d\varepsilon}{2\pi}.\end{aligned} \tag{13}$$

Hence,

$$\begin{aligned}G' &= GVG+FV^*F+iG\Delta' F+iF\Delta'^*G,\\ F' &= DV^*F+FVG+iF\Delta' F-iD\Delta'^*G,\end{aligned} \tag{14}$$

where

$$D=\frac{\varepsilon-H}{\varepsilon^2-H^2-\Delta^2},\qquad G=\frac{\varepsilon+H}{\varepsilon^2-H^2-\Delta^2},\qquad F=\frac{-i\Delta}{\varepsilon^2-H^2-\Delta^2}.$$

Integrating the first of eqs. (14) over $\varepsilon$ in accord with (5) we obtain a first order correction to the density matrix. After simple algebraic calculations we obtain

$$\rho'_{\lambda\lambda'}=\int G'_{\lambda\lambda'}\frac{d\varepsilon}{2\pi i}=\frac{(\varepsilon_\lambda\varepsilon_{\lambda'}-E_\lambda E_{\lambda'})V_{\lambda\lambda'}-\Delta^2V^*_{\lambda\lambda'}+\Delta(\varepsilon_\lambda\Delta'_{\lambda\lambda'}+\varepsilon_{\lambda'}\Delta'^*_{\lambda\lambda'})}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}. \tag{15}$$

In a similar way one can find the quantity

$$\int F'_{\lambda\lambda'}\frac{d\varepsilon}{2\pi}=\frac{\Delta(\varepsilon_\lambda V^*_{\lambda\lambda'}+\varepsilon_{\lambda'}V_{\lambda\lambda'})+\Delta^2\Delta'_{\lambda\lambda'}-(E_\lambda E_\lambda+\varepsilon_\lambda\varepsilon_{\lambda'})\Delta'^*_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}. \tag{16}$$

The last of equations in (13) can be written as

$$\Delta'^*(\mathbf{r})=\gamma\sum_{\lambda\lambda'}\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})\int F'_{\lambda\lambda'}(\varepsilon)\frac{d\varepsilon}{2\pi}.$$

The effect of the perturbation on $\gamma$ has been neglected here. With the help of (12) the left hand side of this equation can be represented in the form

$$\Delta'^*(\mathbf{r})=-\gamma\sum_{\lambda\lambda'}\Delta'^*_{\lambda\lambda'}\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})\frac{1}{2E_\lambda}=-\gamma\sum_{\lambda\lambda'}\Delta'^*_{\lambda\lambda'}\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})\frac{1}{2E_\lambda}.$$

We obtain an integral equation

$$\sum_{\lambda\lambda'}\left[\int F'_{\lambda\lambda'}(\varepsilon)\frac{d\varepsilon}{2\pi}+\frac{\Delta'^*_{\lambda\lambda'}}{4E_\lambda}+\frac{\Delta'^*_{\lambda\lambda'}}{4E_{\lambda'}}\right]\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})=0.$$

Applying (16) we find

$$\sum_{\lambda\lambda'}\frac{2\Delta(\varepsilon_\lambda V_{\lambda\lambda'}+\varepsilon_{\lambda'}V_{\lambda\lambda'})+2\Delta^2\Delta'_{\lambda\lambda}+[2\Delta^2+(\varepsilon_\lambda-\varepsilon_{\lambda'})^2]\Delta'^*_{\lambda\lambda'}}{E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})=0. \tag{17}$$

## 3. Method of Calculation of the Moments of Inertia

### 3.1. BASIC FORMULAE

In order to evaluate the moments of inertia one must determine the variation in the Hamiltonian occurring upon transition to a rotating coordinate system.

Let $H = \varepsilon(\mathbf{p})+U(\mathbf{r})$, where $\mathbf{p}$ is the momentum operator. In order to find the change in $H$ upon passing to a rotating system of coordinates, $\mathbf{p}$ should be replaced by $\mathbf{p}-M(\mathbf{r}\times\Omega)$, where $M$ is the nucleon mass and $\Omega$ is the angular velocity of the coordinate system.

We get

$$V = -\frac{\partial\varepsilon}{\partial\mathbf{p}} M(\mathbf{r}\times\Omega) = \frac{M}{M_{\text{eff}}}\mathcal{M}\cdot\Omega = \frac{M}{M_{\text{eff}}}\mathcal{M}^x\Omega \tag{18}$$

where $\mathcal{M} = \mathbf{r}\times\mathbf{p}$, $M_{\text{eff}} = p/(\partial\varepsilon/\partial p)$ is the effective quasiparticle mass and the $x$-axis is chosen along $\Omega$. Noting that $V^* = -V$ and denoting $\Delta' = if(\mathbf{r})\Omega(M/M_{\text{eff}})$ we obtain from (15)

$$\frac{\rho'_{\lambda\lambda'}}{\Omega} = \frac{\dfrac{M}{M_{\text{eff}}}\left([E_\lambda E_{\lambda'}-\varepsilon_\lambda\varepsilon_{\lambda'}-\Delta^2]\mathcal{M}^x_{\lambda\lambda'}+i\Delta(\varepsilon_\lambda-\varepsilon_{\lambda'})f_{\lambda\lambda'}\right)}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})} \tag{19}$$

where

$$f_{\lambda\lambda'} = \int \varphi_\lambda^*(\mathbf{r})f(\mathbf{r})\varphi_{\lambda'}(\mathbf{r})d\mathbf{r}.$$

Multiplying (17) by $i$ we get

$$\sum_{\lambda\lambda'} \frac{2\Delta\dot{\mathcal{M}}^x_{\lambda\lambda'}+(\varepsilon_\lambda-\varepsilon_{\lambda'})^2 f_{\lambda\lambda'}}{E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r}) = 0. \tag{20}$$

We now determine the mean value of the angular momentum

$$\langle\mathcal{M}^x\rangle = \operatorname{tr}\rho\mathcal{M}^x = \sum_{\lambda\lambda'}\mathcal{M}^x_{\lambda\lambda'}\rho_{\lambda'\lambda}.$$

The moment of inertia can be derived from the expression

$$\mathcal{I} = \frac{\langle\mathcal{M}\rangle}{\Omega}\frac{M}{M_{\text{eff}}}\sum_{\lambda\lambda'}\frac{(E_\lambda E_{\lambda'}-\varepsilon_\lambda\varepsilon_{\lambda'}-\Delta^2)|\mathcal{M}^x_{\lambda\lambda'}|^2-\Delta f_{\lambda\lambda'}\mathcal{M}^x_{\lambda\lambda'}}{2E_\lambda E_\lambda(E_\lambda+E_{\lambda'})}, \tag{21}$$

where $f$ is defined by the integral equation (20).

### 3.2. OTHER DEDUCTION

It may be mentioned that integral equation (20) for $f$ can be deduced in a different way. With the aid of $\rho$ we express the mean value of the current density as $\langle\mathbf{j}(\mathbf{r})\rangle = \operatorname{tr}(\rho'\mathbf{j}_0-\rho_0\mathbf{r}\times\Omega)$. Eq. (20) can then be derived from the condition $\operatorname{div}\langle\mathbf{j}(\mathbf{r})\rangle = 0$ without taking recourse to the third of eqs. (8).

This circumstance permits one to solve the first two equations in (8) with $\Delta = \text{const}$ for an arbitrary initial Hamiltonian without arriving at a

contradiction, even though the last of eqs. (8) yield $\Delta = \text{const}$ only for a rectangular potential well.

### 3.3. PRELIMINARY TRANSFORMATION FOR LARGE LEVEL DENSITY

Consider the expression

$$\mathcal{I}_1 = \sum_{\lambda\lambda'} \frac{(E_\lambda E_{\lambda'} + \varepsilon_\lambda \varepsilon_{\lambda'} - \Delta^2)}{2E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})} |\mathcal{M}^x_{\lambda\lambda'}|^2 \frac{M}{M_{eff}}. \tag{22}$$

It is not difficult to see that the quantity

$$\mathcal{L}(\varepsilon_\lambda \varepsilon_{\lambda'}) = \frac{E_\lambda E_{\lambda'} - \varepsilon_\lambda \varepsilon_{\lambda'} - \Delta^2}{2E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})} \tag{23}$$

as a function of $\varepsilon_\lambda$ for a fixed difference $d = \varepsilon_\lambda - \varepsilon_{\lambda'}$ has a sharp peak with a width $\approx \Delta$ at the point $\varepsilon_\lambda + \varepsilon_{\lambda'} = 0$. Let us denote $\int \mathcal{L}(\varepsilon_\lambda, \varepsilon_\lambda + d) d\varepsilon_\lambda = U(d/2\Delta)$.

It can easily be shown that

$$U(x) = 1 - \frac{\ln(x + \sqrt{1+x^2})}{x\sqrt{1+x^2}}. \tag{24}$$

If a sufficient number of levels is confined within a width $\Delta$, the function $\mathcal{L}$ in (22) can be replaced by

$$\mathcal{L}(\varepsilon_\lambda \varepsilon_{\lambda'}) \to U\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right) \delta(\varepsilon_\lambda). \tag{25}$$

Employing (24) and (25) we obtain instead of (22)

$$\mathcal{I}_1 = \sum_{\lambda\lambda'} \left\{1 - g\left(\frac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right)\right\} |\mathcal{M}^x_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda) \frac{M}{M_{eff}} \tag{26}$$

where

$$g(x) = \frac{\operatorname{arg\,sh} x}{x\sqrt{1+x^2}}. \tag{27}$$

### 3.4. FURTHER DISCUSSION: FIRST TERM

We shall show that the expression

$$\mathcal{I}_0 = \sum_{\lambda\lambda'} |\mathcal{M}^x_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda) \frac{M}{M_{eff}} \tag{28}$$

differs only by a factor from the value of the moment of inertia of a rigid body. Summing over $\lambda'$ in (28) and denoting

$$\rho(\varepsilon_0 \mathbf{r}) = \sum_\lambda \varphi_\lambda^*(\mathbf{r}) \varphi_\lambda(\mathbf{r}) \delta(\varepsilon_\lambda) \tag{29}$$

where $\rho(\varepsilon_0 \mathbf{r})$ is the density of particles with an energy $\varepsilon_0$, we obtain

$$\mathcal{I}_0 = \int d\mathbf{r} \rho(\varepsilon_0 \mathbf{r}) \langle \mathcal{M}^{x2} \rangle \frac{M}{M_{eff}}.$$

Instead of $\mathcal{M}^{x2}$ we have here the value of $\langle \mathcal{M}_x^2 \rangle$ averaged over the directions of the momentum $\mathbf{p}$. Since all directions of $\mathbf{p}$ are equally probable we obtain, after averaging the operator

$$\mathcal{M}_x^2 = y^2 p_z^2 + z^2 p_y^2 - y p_z z p_y - z p_y y p_z$$

over the angles of $\mathbf{p}$,

$$\langle \mathcal{M}_x^2 \rangle = \tfrac{1}{3} p_0^2(\mathbf{r})(y^2+z^2).$$

Denote

$$n'(\mathbf{r}) = \rho(\varepsilon_0 \mathbf{r}) \frac{p_0^2(\mathbf{r})}{3M_{\text{eff}}}.$$

Then

$$\mathcal{I}_0 = \int n'(\mathbf{r})(y^2+z^2)\mathrm{d}\mathbf{r} M.$$

In the quasi-classical approximation the particle density

$$n(\mathbf{r}) = c p_0^3(\mathbf{r}), \qquad p_0 = \sqrt{2M_{\text{eff}}(\varepsilon_0 - U(\mathbf{r}))},$$
$$\rho(\varepsilon_0 \mathbf{r}) = \frac{\partial n(\mathbf{r})}{\partial \varepsilon_0} = 3M_{\text{eff}} \frac{n(\mathbf{r})}{p_0^2(\mathbf{r})}. \tag{30}$$

Thus $n'(\mathbf{r}) = n(\mathbf{r})$ and hence $\mathcal{I}_0$ is identical with the moment of inertia for a rigid body

$$\mathcal{I}_0 = \int M n(\mathbf{r})(y^2+z^2)\mathrm{d}\mathbf{r} = \mathcal{I}_{\text{rig}}.$$

However, owing to the existence of shells the level density near the Fermi surface, $\rho_0 = \int \rho(\varepsilon_0 \mathbf{r})\mathrm{d}\mathbf{r}$, may differ from the classical value which is equal to

$$\rho_0^{\text{cl}} = \int \frac{\partial n(\varepsilon_0 \mathbf{r})}{\partial \varepsilon_0} \mathrm{d}\mathbf{r}$$

and hence

$$\mathcal{I}_0 = \frac{\rho_0}{\rho_0^{\text{cl}}} \mathcal{I}_{\text{rig}}. \tag{31}$$

It should be noted that the effective mass (as one may have expected) does not enter $\mathcal{I}_0$.

For sufficiently deformed nuclei ($\beta \gtrsim A^{-\frac{1}{3}}$), $\rho_0/\rho_0^{\text{cl}} \approx 1$ and $\mathcal{I}_0 = \mathcal{I}_{\text{rig}}$. For small deformations, $\beta \ll A^{-\frac{1}{3}}$, when the shell structure significantly influences the level density, $\rho_0/\rho_0^{\text{cl}} > 1$.

### 3.5. FURTHER DISCUSSION: SECOND TERM

Consider now the second term in the moment of inertia

$$\mathcal{I}_2 = -\Delta \sum_{\lambda\lambda'} \frac{f_{\lambda\lambda'} \dot{\mathcal{M}}^x_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})} \frac{M}{M_{\text{eff}}}.$$

Multiplying eq. (20) by $f(\mathbf{r})$ and integrating over $\mathbf{r}$ we get

$$-\Delta \sum_{\lambda\lambda'} \frac{\dot{\mathcal{M}}^z_{\lambda\lambda'} f_{\lambda'\lambda}}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})} = \sum_{\lambda\lambda'} \frac{(\varepsilon_\lambda-\varepsilon_{\lambda'})^2 |f_{\lambda\lambda'}|^2}{4E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}.$$

Thus

$$\mathcal{J}_2 = \sum_{\lambda\lambda'} \frac{(\varepsilon_\lambda-\varepsilon_{\lambda'})^2 |f_{\lambda\lambda'}|^2}{4E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})} \frac{M}{M_{\text{eff}}}.$$

Making use of the sharp peak in $1/E_\lambda E_{\lambda'}(E_\lambda+E_\lambda)$ as a function of $\varepsilon_\lambda$ as was done in the computation of $\mathcal{J}_1$ we obtain

$$\mathcal{J}_2 = \frac{M}{M_{\text{eff}}} \sum_{\lambda\lambda'} |f_{\lambda\lambda'}|^2 g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)^2 \delta(\varepsilon_\lambda), \tag{32}$$

where $g(x)$ is defined by expression (27).

### 3.6. EXPRESSION FOR THE MOMENT OF INERTIA

Exploiting the sharp maximum of $1/E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})$ we obtain instead of (20)

$$\sum_{\lambda\lambda'} \left[\frac{\dot{\mathcal{M}}^z_{\lambda\lambda'}}{2\Delta} + \left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)^2 f_{\lambda\lambda}\right] g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right) \varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r})\delta(\varepsilon_\lambda) = 0. \tag{33}$$

Thus, the moment of inertia can be calculated by the formula

$$\mathcal{J} = \mathcal{J}_0 - \frac{M}{M_{\text{eff}}} \sum_{\lambda\lambda'} g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right) |\mathcal{M}^z_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda) + \frac{M}{M_{\text{eff}}} \sum_{\lambda\lambda'} g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right) \times \left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)^2 |f_{\lambda\lambda'}|^2 \delta(\varepsilon_\lambda), \tag{34}$$

in which $f(\mathbf{r})$ should be inserted as determined from integral equation (33).

We introduce the quantity

$$x = \frac{\varepsilon_0 \beta}{\Delta p_0 R_0}, \tag{35}$$

where $\beta = 2(a-b)/(a+b)$ is a nuclear deformation parameter and $a$ and $b$ are the spheroid semi-axes ($\hbar = 1$).

It can easily be shown that, for $x \gg 1$,

$$\mathcal{J} \to \mathcal{J}_0. \tag{36}$$

In another limiting case, when $x/\beta \ll 1$, one obtains, assuming that the potential $U$ has the form

$$U\left(\frac{z^2}{a^2} + \frac{x^2+y^2}{b^2}\right)$$

$$\mathcal{J} = \mathcal{J}_0 \frac{a^2-b^2}{a^2+b^2} + C_1 \mathcal{J}_0 x^2, \tag{37}$$

where $C_1 \approx 1$.

Thus, for $x/\beta \ll 1$, the moment of inertia of the system approaches that of an ideal fluid. This result is quite that to be expected. The quantity $x/\beta$ is of the order of $a_0/R$ where $a_0 = p_0/\Delta M$ is the correlation radius of the paired particles. If the correlation radius is much smaller than the nuclear radius the hydrodynamical equations for an ideal fluid should be applicable.

For $\beta \to 0$ and a fixed value of $\Delta$, one has $\mathcal{I} \approx \beta^2$.

## 4. Calculation of the Moment of Inertia for an Oscillator Potential

### 4.1. CALCULATION OF $\mathcal{I}_1$

The potential $U(\mathbf{r})$ will be written down as

$$U(\mathbf{r}) = \tfrac{1}{2}M_{\text{eff}}[\omega_z^2 z^2 + \omega_y^2(x^2+y^2)]. \tag{38}$$

We have the operator relation

$$\dot{\mathcal{M}}_x = z\frac{\partial U}{\partial y} - y\frac{\partial U}{\partial z} = (\omega_y^2 - \omega_z^2)yzM_{\text{eff}}.$$

Thus

$$\dot{\mathcal{M}}^x_{\lambda\lambda'} = (\omega_y^2 - \omega_z^2)y_{mm'}z_{nn'}M_{\text{eff}}.$$

The matrix element $\dot{\mathcal{M}}^x_{\lambda\lambda'}$ therefore is only non-vanishing when $m' = m \pm 1$, $n' = n \pm 1$. With quasi-classical accuracy ($n \gg 1$, $m \gg 1$) all four values of $\dot{\mathcal{M}}^x_{\lambda\lambda'}$ are identical. Noting, moreover, that for the corresponding values of $\varepsilon_\lambda - \varepsilon_{\lambda'} = \pm(\omega_z - \omega_y), \pm(\omega_z + \omega_y)$ we find from (26)

$$\begin{aligned}\mathcal{I} &= \mathcal{I}_0 - \frac{M}{M_{\text{eff}}}\sum_{\lambda\lambda'}\frac{g\left(\dfrac{\varepsilon_\lambda - \varepsilon_{\lambda'}}{2\Delta}\right)|\dot{\mathcal{M}}^x_{\lambda\lambda'}|^2\delta(\varepsilon_\lambda)}{(\varepsilon_\lambda - \varepsilon_{\lambda'})^2} \\ &= \mathcal{I}_0 - \frac{M}{2M_{\text{eff}}}\left(\frac{g_1}{v_1} + \frac{v_2}{v_2}\right)\sum_{\lambda\lambda'}\frac{|\dot{\mathcal{M}}^x_{\lambda\lambda'}|^2\delta(\varepsilon_\lambda)}{4\Delta^2},\end{aligned} \tag{39}$$

where

$$v_1 = \frac{\omega_z - \omega_y}{2\Delta}, \qquad v_2 = \frac{\omega_z + \omega_y}{2\Delta}, \qquad g_1 = g(v_1), \qquad g_2 = g(v_2).$$

Similarly,

$$\mathcal{I}_0 = \sum |\dot{\mathcal{M}}^x_{\lambda\lambda'}|^2\delta(\varepsilon_\lambda)\frac{M}{M_{\text{eff}}} = \tfrac{1}{2}\frac{(v_1^2+v_2^2)}{v_1^2v_2^2}\sum|\dot{\mathcal{M}}^x_{\lambda\lambda'}|^2\delta(\varepsilon_\lambda)\frac{M}{M_{\text{eff}}}\frac{1}{4\Delta^2}.$$

Inserting in (39) we obtain

$$\mathcal{I}_1 = \mathcal{I}_0\left(1 - \frac{g_1v_2^2 + g_2v_1^2}{v_1^2+v_2^2}\right). \tag{40}$$

### 4.2. CALCULATION OF $\mathscr{I}_2$

We shall now show that eq. (33) is satisfied for $f = \alpha yz$. As above we make use of the fact that for prescribed value of $\lambda$ matrix elements $\dot{\mathscr{M}}^z_{\lambda\lambda'}$, $f_{\lambda\lambda'}$ and $\varphi_\lambda\varphi_{\lambda'}$ are independent of $\lambda'$.

Eq. (33) yields

$$\Delta(g_1+g_2)\sum \dot{\mathscr{M}}^z_{\lambda\lambda'}\varphi_\lambda\varphi^*_{\lambda'}\delta(\varepsilon_\lambda) = -2\Delta^2(g_1\nu_1^2+g_2\nu_2^2)\sum f_{\lambda\lambda'}\varphi_\lambda\varphi^*_{\lambda'}\delta(\varepsilon_\lambda).$$

Summing over $\lambda'$ we get

$$f(\mathbf{r}) = -\frac{g_1+g_2}{2\Delta(g_1\nu_1^2+g_2\nu_2^2)}\dot{\mathscr{M}}^z = -\frac{(g_1+g_2)(\omega_y^2-\omega_z^2)yz}{2\Delta(g_1\nu_1^2+g_2\nu_2^2)}M_{\text{eff}}. \quad (41)$$

Inserting (41) in (32) we find

$$\mathscr{I}_2 = \mathscr{I}_0\frac{(g_1+g_2)^2\nu_1^2\nu_2^2}{(\nu_1^2+\nu_2^2)(\nu_1^2g_1+\nu_2^2g_2)}. \quad (42)$$

It can readily be seen that for $\nu_1, \nu_2 \gg 1$, expression (40) goes over to $\mathscr{I}_0$ whereas $\mathscr{I}_2$, defined by (42), tends to zero so that the asymptotic formula (36) is satisfied.

Relation (37) holds for $\nu_1, \nu_2 \ll 1$, that is, the moment of inertia approaches the moment of inertia for an ideal fluid.

### 4.3. MOMENT OF INERTIA

If numerical values of $\nu_1$ and $\nu_2$ for nuclei are inserted, e.g. $\nu_1 \approx 1$, $\nu_2 \approx 10$ the expression for $\mathscr{I}$ can be written in the form

$$\mathscr{I} \approx \mathscr{I}_0\left\{1-g_1+\frac{g_1^2\nu_1^2}{g_1\nu_1^2+\ln 2\nu_2}\right\} = \mathscr{I}_0\phi_1(\nu_1). \quad (43)$$

The values of function $\phi_1(x)$ for $\nu_2 = 10$ are presented in table 1.

TABLE 1

Values of functions $\phi_1$ and $\phi_2$ defined by formulas (43) and (51)

| $x$ | $\phi_1(x)$ | $\phi_2(x)$ | $x$ | $\phi_1(x)$ | $\phi_2(x)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1.3 | 0.64 | 0.24 |
| 0.1 | 0.01 | 0.005 | 1.4 | 0.67 | 0.26 |
| 0.2 | 0.03 | 0.02 | 1.5 | 0.71 | 0.28 |
| 0.3 | 0.07 | 0.03 | 1.6 | 0.74 | 0.30 |
| 0.4 | 0.13 | 0.06 | 1.7 | 0.75 | 0.32 |
| 0.5 | 0.19 | 0.08 | 1.8 | 0.77 | 0.34 |
| 0.6 | 0.26 | 0.10 | 1.9 | 0.79 | 0.36 |
| 0.7 | 0.32 | 0.13 | 2.0 | 0.80 | 0.37 |
| 0.8 | 0.38 | 0.15 | 2.2 | 0.81 | 0.40 |
| 0.9 | 0.43 | 0.17 | 2.4 | 0.83 | 0.43 |
| 1.0 | 0.49 | 0.19 | 2.6 | 0.85 | 0.45 |
| 1.1 | 0.53 | 0.21 | 2.8 | 0.86 | 0.48 |
| 1.2 | 0.60 | 0.23 | 2.0 | 0.87 | 0.50 |

## 5. Calculation of the Moment of Inertia for a Rectangular Potential Well

### 5.1. REDUCTION OF EXPRESSION FOR MOMENT OF INERTIA

From expressions (26) and (28) we have

$$\mathcal{I}_0-\mathcal{I}_1=\sum g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)\mathcal{M}^x_{\lambda\lambda'}|^2\delta(\varepsilon_\lambda)\frac{M}{M_{\text{eff}}}. \tag{44}$$

We shall show that only those values of $\lambda'$ should be retained in this sum for which $m'=m\pm1$, $\nu'=\nu$, where $m$ is the eigenvalue of the angular momentum projection on the symmetry axis and $\nu$ is the set of the remaining quantum numbers.

The relationship $m'=m\pm1$ is an exact selection rule for the matrix elements of operator $\mathcal{M}^*$. We shall show that terms $\nu'\neq\nu$ yield a small contribution to sum (44).

The index $\nu$ is determined by three quantum numbers. For a nucleus of almost spherical shape it is defined by the radial quantum number $n$, orbital angular momentum $l$ and total angular momentum $j=l\pm\frac{1}{2}$. If one of these quantum numbers changes, the order of magnitude of the energy change will be $|\varepsilon_\lambda-\varepsilon_{\lambda'}|\approx\varepsilon_0/p_0R\approx\varepsilon_0A^{-\frac{1}{3}}\gg\Delta$, the corresponding value of the function $g\approx(\Delta p_0R/\varepsilon_0)^2\ll1$. Thus, the contribution of such terms to the sum (44) is negligible.

Small differences $|\varepsilon_\lambda-\varepsilon_{\lambda'}|\approx\varepsilon_0A^{-\frac{2}{3}}$ can appear as a result if an increase of some quantum number by several units and simultaneous decrease of another. In this case $g\approx1$, but the matrix element $\mathcal{M}^x_{\lambda\lambda'}$ will be small owing to the large difference in the number of nodes of functions $\varphi_\lambda$ and $\varphi_{\lambda'}$; moreover, it will contain as a factor $\beta$, since for a spherical box $\mathcal{M}^x_{\lambda\lambda'}=0$ if $\nu'\neq\nu$. Thus, in the sum (44) only terms with $\nu=\nu'$ and $m'=m\pm1$ should be retained, and

$$\mathcal{I}_0-\mathcal{I}_1\approx\sum g\left[\frac{\varepsilon_{\nu,m+1}-\varepsilon_{\nu,m}}{2\Delta}\right]\{|\mathcal{M}^x_{\substack{\nu,m\\\nu,m+1}}|^2+|\mathcal{M}^x_{\substack{\nu,m\\\nu,m-1}}|^2\}\delta(\varepsilon_{\nu m})\frac{M}{M_{\text{eff}}}. \tag{45}$$

### 5.2. COMPUTATION OF MOMENT OF INERTIA

The following calculations will be performed for a spherical box of effective radius $R_0$. The quantity $\mathcal{I}/\mathcal{I}_0$ is a symmetric function of the semiaxes $a$ and $b$; this can be seen, in particular, from the calculations for an oscillator potential. Thus, choosing in the final result a radius $R_0$ equal to any mean value symmetric with respect to $a$ and $b$ we can determine $\mathcal{I}/\mathcal{I}_0$ with a small error of the order of $\beta^2$.

For the functions of a spherical box we have

$$|\mathcal{M}^x_{\substack{\nu,m\\\nu,m+1}}|^2+|\mathcal{M}^x_{\substack{\nu,m\\\nu,m-1}}|^2=\tfrac{1}{2}(j^2-m^2)+\tfrac{1}{4}j\approx\tfrac{1}{2}(l^2-m^2), \tag{46}$$

since in the sum (45) large values of $j$ are important and hence $j = l \pm \frac{1}{2}$ can be replaced by $l$. The dependence of the particle energy on $m$ is given by the equation

$$\varepsilon_{\nu m} = \varepsilon_\nu + \beta[(m^2/l^2) - \tfrac{1}{3}](\varepsilon_0 + \varepsilon_\nu), \tag{47}$$

where $\varepsilon_\nu$ is the energy (relative to $\varepsilon_0$ )of a particle in the spherical box.

Expression (47) can readily be obtained by perturbation theory methods. Performing the coordinate transformation which changes an ellipsoid into a sphere we obtain in the first order in $\beta$ the following perturbation operator:

$$Q = \tfrac{1}{3}\beta(p^2 - 3p_z^2)/M_{\text{eff}}.$$

Calculating the energy by the formula

$$\varepsilon_{\nu m} = \int \varphi_\lambda^* Q \varphi_\lambda \mathrm{d}\mathbf{r} + \varepsilon_\nu,$$

where $\varphi_\lambda$ are the eigenfunctions in a spherical box of radius $R_0$, we arrive at formula (47).

Insertion of (46) and (47) in (45) yields

$$\begin{aligned} \mathscr{I}_0 - \mathscr{I}_1 &= \sum_{\nu, m} g \left(\frac{\beta m \varepsilon_0}{\Delta l^2}\right) \tfrac{1}{2}(l^2 - m^2)\delta(\varepsilon_\nu) \frac{M}{M_{\text{eff}}} \\ &= \sum_{l, m} g \left(\frac{\beta m \varepsilon_0}{\Delta l^2}\right) \tfrac{1}{2}(l^2 - m^2)\rho_{lm}(\varepsilon_0) \frac{M}{M_{\text{eff}}}, \end{aligned} \tag{48}$$

where $\rho_{lm}(\varepsilon_0)$ is the energy density of levels possessing prescribed values of $l$ and $m$. By evaluating the volume of phase space corresponding to given values of $l$ and $m$, it is not difficult to obtain (as in the Thomas-Fermi theory of the atom)

$$\rho_{lm}(\varepsilon_0) = \tfrac{3}{2}\rho_0 \frac{\sqrt{(l_0^2 - l^2)}}{l_0^3}, \tag{49}$$

where $l_0 = p_0 R_0$. Applying (30) and (31) we obtain

$$\mathscr{I}_0 = \tfrac{2}{5} N M R_0^2 = \tfrac{2}{15}\rho_0 l_0^2 \frac{M}{M_{\text{eff}}}. \tag{50}$$

Inserting (49) and (50) in (48) and introducing the variables $\eta = m/l_0$, $\xi = l/l_0$, we get

$$\mathscr{I}_1 = \mathscr{I}_0 \left\{1 - \tfrac{45}{4}\int_0^1 \mathrm{d}\xi \xi^3 \sqrt{(1-\xi^2)} \int_0^1 \mathrm{d}\eta (1-\eta^2) g\left(\frac{x\eta}{\xi}\right)\right\} = \mathscr{I}_0 \phi_2(x), \tag{51}$$

where $x$ is defined by expression (35).

The values of function $\phi_2(x)$ are presented in table 1.

5.3. LIMITING CASES

In the case of a rectangular potential well the solution of eq. (33) can be found in analytical form only for the limiting cases $x \ll 1$ and $x \gg 1$. The quantity $\mathcal{I}_2$ turns out to be much smaller than $\mathcal{I}_1$ and hence a solution of (33) of this type is of no practical interest.

Without giving the deduction we present the expressions for $\mathcal{I}_2$ when $x \ll 1$ and $x \gg 1$:

$$\begin{aligned} \mathcal{I}_2 &\underset{x\to 0}{\approx} \tfrac{1}{6}\mathcal{I}_0 \frac{x^2}{\ln(5\varepsilon_0/\Delta p_0 R_0)}, \\ \mathcal{I}_2 &\underset{x\to \infty}{\approx} \tfrac{5}{21}\mathcal{I}_0 \frac{\ln^3 a_1 x}{x^2 \ln(a_2\varepsilon_0/\Delta p_0 R_0)}, \end{aligned} \tag{52}$$

where $a_1$ and $a_2$ are numbers of the order of unity.

It can be concluded from expression (52) that $\mathcal{I}_2/\mathcal{I}_0 \lesssim 5\,\%$ and hence $\mathcal{I} \approx \mathcal{I}_1$.

## 6. Neutron and Proton Moments of Inertia. The Gyromagnetic Ratio

6.1. TWO-FLUID MODEL

The calculations presented above refer to a single type of particle. In order to compare with the experiments both types of nuclear particles should be taken into account.

For $Z > 20$ the neutron and proton Fermi surfaces move apart and neutron-proton pairing becomes impossible. Thus, when $Z-20 \gg 1$, there are two fluids, a neutron and a proton fluid, and they are incapable of exchanging angular momentum as the excitations in each of them have a gap. The nuclear moment of inertia is therefore equal to the sum of the neutron and proton moments of inertia. Denoting by $\phi(x)$ the ratio $\mathcal{I}/\mathcal{I}_0 = \phi(x)$ calculated by us above we obtain the following expression for the moment of inertia:

$$\frac{\mathcal{I}}{\mathcal{I}_0} = \frac{N}{A}\phi(x_n) + \frac{Z}{A}\phi(x_p), \tag{53}$$

where $x_n$ and $x_p$ are the neutron and proton values.

6.2. GYROMAGNETIC RATIOS

The quantities $\Delta_n$ and $\Delta_p$ which enter $x_n$ and $x_p$ cannot be computed theoretically and must be taken from experiment. It will be shown below that $\Delta_n$ is somewhat smaller than $\Delta_p$ and therefore only $\Delta_n$ can be determined from the first single particle levels. The quantity $\Delta_p$ can be determined from the experimental gyromagnetic ratio for rotating nuclei. The gyromagnetic ratio turns out to be much smaller than $Z/A$ †. For a rotating nucleus it is

† My attention was drawn to this fact by D. F. Zaretsky.

evidently

$$g_r = \frac{\int \mathbf{r}\times\mathbf{j}_p d\mathbf{r}}{\int \mathbf{r}\times(\mathbf{j}_p+\mathbf{j}_n)d\mathbf{r}} = \frac{\mathcal{I}_p}{\mathcal{I}_n+\mathcal{I}_p} = \frac{Z\phi(x_p)}{N\phi(x_n)+Z\phi(x_p)}. \qquad (54)$$

The measured value of $g_r$ [6]) for the $Sm^{152}$, $Sm^{154}$ and $Nd^{150}$ nuclei is $g_r \approx$ $\approx 0.21\pm0.04$, whence $\mathcal{I}_n/\mathcal{I}_p = N^0 p(x_n)/Z\phi(x_p) \approx 3.5\pm1$.

In the interval $x = 1$ to 2 the function $\phi_2$ can be represented quite accurately in the form $\phi_2 = Cx$, whence we obtain

$$\frac{\mathcal{I}_n}{\mathcal{I}_p} = \left(\frac{N}{Z}\right)^{\frac{1}{3}} \frac{\Delta_p}{\Delta_n} \approx 3.5\pm1.0,$$

which yields $\Delta_p/\Delta_n \approx 1.5\div2.5$ $\phi_1(x)$ gives a somewhat larger value. This value of $\Delta_p/\Delta_n$ agrees within the experimental errors with the value $\Delta_p/\Delta_n$ derived from the mass defects in this part of the table ($\Delta_p/\Delta_n = 1.5\pm0.3$).

## 7. Moments of Inertia of Odd-Mass Nuclei. Moments of Inertia in Excited States

### 7.1. FLUCTUATIONS IN MOMENTS OF INERTIA

Integrating expression (12) over the volume and denoting $\gamma/V = \gamma_1$, we get

$$1 = -\gamma_1 \sum_\lambda \frac{1}{2E_\lambda}. \qquad (12')$$

The quantity $\gamma_1$ is entirely determined by the density of nuclear matter and nuclear forces and should not appreciably vary from nucleus to nucleus.

We shall now show that the $\Delta$ determined from (12′) experiences significant fluctuations.

We write (12′) in the form

$$1 = -\tfrac{1}{2}\gamma_1 \int_{-\varepsilon_0}^{\varepsilon_1} \frac{\rho(\varepsilon)d\varepsilon}{\sqrt{\Delta^2+\varepsilon^2}} \qquad (12'')$$

where $\varepsilon_1 \approx \varepsilon_0$, and $\rho(\varepsilon)$ is the energy density of the levels.

If, upon transition from one nucleus to another, the level density changes by $\delta\rho$ near the Fermi surface, the corresponding change of $\Delta$, as can be seen from (12″), will be given by the relation

$$\tfrac{1}{2}\int_{-\varepsilon_0}^{\varepsilon_1} \frac{\delta\rho d\varepsilon}{\sqrt{\Delta^2+\varepsilon^2}} - \int_{-\varepsilon_0}^{\varepsilon_1} \rho(\varepsilon)\Delta\delta\Delta \frac{d\varepsilon}{\sqrt{(\Delta^2+\varepsilon^2)^3}} = 0;$$

and since

$$\tfrac{1}{2}\int \Delta^2 \frac{d\varepsilon}{\sqrt{(\Delta^2+\varepsilon^2)^3}} = 1,$$

we get

$$\frac{\delta\Delta}{\Delta} = \tfrac{1}{2}\int \delta\rho \frac{d\varepsilon}{\rho_0\sqrt{\Delta^2+\varepsilon^2}}. \tag{55}$$

It is natural to expect such fluctuations in $\rho$ that $\int\delta\rho d\varepsilon \approx 1$, and hence $\delta\Delta/\Delta \approx 1/(3\rho_0\Delta) \approx 10\%$ if one assumes $\rho_0 = \frac{3}{2}N/\varepsilon_0 \approx 4$ (in the rare earth region )and $\Delta = 0.8$ MeV. This estimate agrees satisfactorily with the fluctuation values of $\Delta$ found from the mass defects or from the energy of the first levels.

Fluctuations of $\Delta$ together with fluctuations of the sum (34) defining $\mathcal{I}/\mathcal{I}_0$ lead to appreciable fluctuations in the moment of inertia $\delta\mathcal{I}/\mathcal{I} \approx 20\%$.

### 7.2. SYSTEMATIC INCREASE OF MOMENT OF INERTIA

Along with fluctuations leading to a chaotic variation of the moment of inertia there is a systematic decrease of $\Delta$ and hence increase of the moment of inertia upon transition to an odd-mass element.

Consider the excited states of the system.

It can be shown that the equation for $\Delta$ changes as follows [5]):

$$1 = -\gamma_1 \sum_\lambda \frac{(1-2n_\lambda)}{2E_\lambda}, \tag{56}$$

where $n_\lambda$ is the number of quasi-particles.

The transition to an odd-mass nucleus should be regarded as the appearance of a quasiparticle with an energy close to $\varepsilon_0$: $n_\lambda = \delta_{\lambda\lambda_1}$, $E_{\lambda_1} \approx \Delta$.

From (56) we obtain, as in (55),

$$\frac{(\Delta'-\Delta'')}{\Delta} = \frac{1}{\rho_0\Delta''}, \tag{57}$$

where $\Delta'$ and $\Delta''$ are the corresponding values for an even and odd-mass nucleus.

The variation of the moment of inertia due to the change in $N$ is

$$\frac{\mathcal{I}''-\mathcal{I}'}{\mathcal{I}'} = \frac{\mathcal{I}''_n-\mathcal{I}'_n}{\mathcal{I}'} = \frac{\mathcal{I}'_n}{\mathcal{I}'} x_n \frac{d\ln\phi(x_n)}{dx_n}\cdot\frac{\Delta'_n-\Delta''_n}{\Delta''_n} = 0.9\frac{\mathcal{I}'_n}{\mathcal{I}'}\cdot\frac{\Delta}{\Delta''^2_n}\cdot\frac{1}{\rho_0{}^n}. \tag{58}$$

This equation was obtained with the help of (57) and the approximate formula

$$\frac{x_n d\ln\phi_{1,2}(x_n)}{dx_n} \approx 0.9.$$

In a similar manner we obtain the change of the moment of inertia upon changing to odd $Z$:

$$\frac{\mathcal{I}''-\mathcal{I}'}{\mathcal{I}'} = \frac{\mathcal{I}''_p-\mathcal{I}'_p}{\mathcal{I}'} = 0.9\frac{\mathcal{I}'_p}{\mathcal{I}'}\cdot\frac{\Delta_p}{\rho_0{}^p\Delta''^2_p}. \tag{59}$$

It will be shown below that expressions (58) and (59) satisfactorily agree with the experimental data.

### 7.3. INCREASE OF MOMENT OF INERTIA IN SINGLE-PARTICLE EXCITATION

An appreciable increase of the moment of inertia should occur during single-particle excitations of nuclei. Thus, during an excitation entailing the formation of one hole and one quasi particle, we find from (56) that the relative decrease of $\Delta$ is twice as large as that for transition from an even to an odd-mass nucleus, and hence the relative increase of the moment of inertia is twice as large.

## 8. Determination of $\Delta$ from Nuclear Masses

From the definition of the Green's function $G$ it is not difficult to deduce that

$$G_\lambda = \sum_S \frac{|(\phi^S_{N+1},\, a_\lambda^+ \phi_N^0)|^2}{\varepsilon - E_S(N+1) + E_0(N) + i\delta} + \sum_S \frac{|(\phi^S_{N-1},\, a_\lambda \phi_N^0)|^2}{\varepsilon + E_S(N-1) - E_0(N) - i\delta}. \quad (60)$$

Here $a_\lambda$, $a_\lambda^+$ are the annihilation and creation operators for particles in a state $\lambda$.

Comparing formula (60) with expression (9) we obtain

$$E_0(N+1) - E_0(N) = \Delta + \varepsilon_0, \quad (61)$$

$$E_0(N) - E_0(N-1) = -\Delta + \varepsilon_0. \quad (62)$$

Formulas (61) and (62) permit one to find $\Delta$ from the binding energy of the nuclei. A more precise expression can be obtained if the dependence on $N$ which is not related to pairing, is eliminated from $E(N)$. For this purpose we set up an expression from which terms containing the first and second derivatives in the expansion in $N'-N$ are eliminated.

This requirement is met by the quantity

$$\tfrac{1}{4}[3E(N+1) - 3E(N) + E(N-1) - E(N+2)] = \Delta. \quad (63)$$

Values of $\Delta'_{n,p}$ and $\Delta''_{n,p}$ found from (63) and (57) are presented in table 2.

TABLE 2

Values of $\Delta_n$ and $\Delta_p$ computed by formulas (57) and (63) from the mass defects of some elements

| Element | $\Delta_n$ (MeV) | $\Delta_p$ (MeV) | Element | $\Delta_n$ (MeV) | $\Delta_p$ (MeV) |
|---|---|---|---|---|---|
| $Sm^{150}$ | 1.32 | 1.64 | $Hf^{178}$ | 0.62 | – |
| $Gd^{154}$ | 1.15 | – | $Hf^{179}$ | 0.42 | – |
| $Gd^{156}$ | 0.92 | – | $Th^{230}$ | 0.76 | 1.05 |
| $Gd^{157}$ | 0.72 | – | $Th^{231}$ | 0.56 | 0.81 |
| $Dy^{162}$ | 0.82 | – | $Th^{232}$ | 0.55 | 0.78 |
| $Dy^{163}$ | 0.62 | – | $U^{238}$ | 0.64 | 0.85 |

## 9. Comparison with Experiment

### 9.1. ESTIMATES OF THEORETICAL PARAMETERS

In order to compare expressions (43) and (51) with the experimental moments of inertia we express $x$ and $\nu_1$ through observable quantities. As mentioned above, for $R_0$ in (35) one should take some mean radius, eg. $R_0 = \frac{1}{2}(a+b)$. We obtain $R_0 = R(1+\frac{1}{3}\beta)$, where $R^3 = ab^2$. Putting $R = 1.2\times 10^{-13} A^{\frac{1}{3}}$ cm, we find

$$\begin{aligned} \varepsilon_0^{\,n} &= 52\left(\frac{N}{A}\right)^{\frac{2}{3}}\frac{M}{M_{\text{eff}}}, \qquad p_0^{\,n} R = 1.9N^{\frac{1}{3}}, \\ x_n &= 27\left[\frac{\beta}{(1+\frac{1}{3}\beta)\Delta_n A^{\frac{1}{3}}}\right]\left(\frac{N}{A}\right)^{\frac{1}{3}}\frac{M}{M_{\text{eff}}}. \end{aligned} \tag{64}$$

Here $\Delta_n$ is expressed in MeV. Similar expressions can be derived for protons.

We choose $\omega_0 = \frac{1}{2}(\omega_y+\omega_z)$ from the requirement of identity of the mean square values of $\langle r^2\rangle$ for a rectangular and oscillator potentials. This yields

$$\langle y^2\rangle = \tfrac{1}{4}\varepsilon_0/\omega_0^{\,2} = \tfrac{1}{5}R_0^{\,2}.$$

Expressions analogous to (64) can easily be derived:

$$\begin{aligned} \varepsilon_0^{\,n} &= 76\left(\frac{N}{A}\right)^{\frac{2}{3}}\frac{M}{M_{\text{eff}}}, \qquad p_0^{\,n} R = 2.3N^{\frac{1}{3}}, \\ \nu_1^{\,n} &= \tfrac{1}{2}\frac{\beta\omega_0}{\Delta_n} = 0.95x_n \approx x_n. \end{aligned} \tag{65}$$

TABLE 3

Values of the moments of inertia, deformation parameters $\beta$ and parameters $x_n$ and $x_p$ defined by (64).

| Element | $\beta$ [7] | $x_p$ | $x_n$ | $\left(\frac{\mathcal{I}}{\mathcal{I}_0}\right)_{\text{rect.}}$ | $\left(\frac{\mathcal{I}}{\mathcal{I}_0}\right)_{\text{osc.}}$ | $\left(\frac{\mathcal{I}}{\mathcal{I}_0}\right)^{[7]}_{\text{exper.}}$ |
|---|---|---|---|---|---|---|
| $Nd^{150}$ | 0.26 | 0.54 | 0.94 | 0.15 | 0.38 | 0.35 |
| $Sm^{152}$ | 0.24 | 0.65 | 1.02 | 0.17 | 0.43 | 0.38 |
| $Gd^{154}$ | 0.26 | 0.52 | 0.88 | 0.13 | 0.35 | 0.36 |
| $Gd^{156}$ | 0.33 | 0.87 | 1.37 | 0.22 | 0.57 | 0.48 |
| $Gd^{157}$ | 0.29 | 0.93 | 1.60 | 0.22 | 0.64 | 0.60 |
| $Dy^{162}$ | 0.30 | 0.84 | 1.43 | 0.23 | 0.57 | 0.50 |
| $Hf^{179}$ | 0.20 | 0.99 | 1.75 | 0.27 | 0.66 | 0.52 |
| $Os^{186}$ | 0.18 | 0.44 | 0.69 | 0.09 | 0.26 | 0.28 |
| $Th^{230}$ | 0.22 | 0.63 | 0.95 | 0.15 | 0.40 | 0.43 |
| $Th^{232}$ | 0.22 | 0.84 | 1.42 | 0.24 | 0.60 | 0.44 |
| $U^{238}$ | 0.24 | 0.83 | 1.29 | 0.22 | 0.54 | 0.43 |

In order to evaluate $x_p$ when the mass defect values of $\Delta_p$ are not available, it is assumed that $\Delta_p/\Delta_n = 1.5$.

$(\mathcal{I}/\mathcal{I}_0)_{\text{osc.}}$ is the calculated value of the moment of inertia for an oscillator potential; $(\mathcal{I}/\mathcal{I}_0)_{\text{rect.}}$ the same for a rectangular potential well.

The effective mass, $M_{eff}$, apparently differs little from the nucleon mass $M$ †.

Values of $x$ for $M/M_{eff} = 1$ are given in table 3 and fig. 1. As can be seen

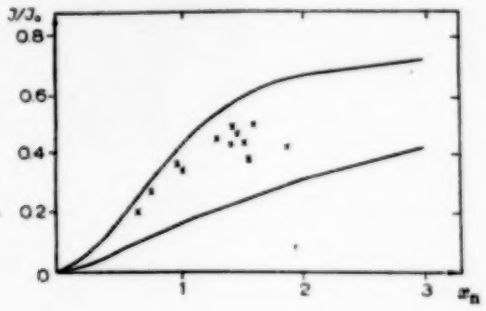


Fig. 1. Experimental values of $\mathscr{I}/\mathscr{I}_0$ as a function of $x_n$. Theoretical curves of $\mathscr{I}/\mathscr{I}_0$ for an oscillating potential (upper curve) and rectangular potential well when $\mathscr{I}_n/\mathscr{I}_p = 2.5$.

from fig. 1 all experimental values of $\mathscr{I}/\mathscr{I}_0$ lie between the two theoretical curves.

In order to compare relations (58) and (59) with the experimental data we evaluate the mean value of the observed moment of inertia over a set of nuclei possessing close values of $N$ and $Z$. In the rare earth region the mean value for 5 elements is

$$\left(\frac{\mathscr{I}}{\mathscr{I}_0}\right)_{\text{even-neutron nuclei}} = 0.42, \quad \left(\frac{\mathscr{I}}{\mathscr{I}_0}\right)_{\text{odd-neutron nuclei}} = 0.53, \quad \frac{\mathscr{I}''-\mathscr{I}'}{\mathscr{I}'} = 0.26.$$

From formula (58) we obtain $(\mathscr{I}''-\mathscr{I}')/\mathscr{I}' = 0.32$. Upon transition to even $N$ and odd $Z$ we obtain from (58) and (59)

$$\frac{(\mathscr{I}''-\mathscr{I}')_{\text{odd } N}}{(\mathscr{I}''-\mathscr{I}')_{\text{odd } Z}} \approx \frac{\mathscr{I}_n \Delta_p}{\mathscr{I}_p \Delta_n} \approx 4.$$

This is in agreement with the fact that a transition to odd-proton nuclei has a much smaller effect on the moment of inertia than a transition to odd-neutron nuclei.

The author is thankful to L. D. Landau, D. F. Zaretsky and A. I. Larkin for interesting discussions and also to I. M. Pavlichenkov and M. G. Urin for compiling the tables.

† The effective mass also enters the expression for the orbital magnetic moment of the nucleus. If the difference between $M_{eff}$ and $M$ were appreciable the nuclear magnetic moments would not lie between the Schmidt curves.

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7.A:8.B *Nuclear Physics* **13** (1959) 675—684; © *North-Holland Publishing Co., Amsterdam*
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# SPECTRAL REPRESENTATIONS FOR SOME VERTEX PARTS

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Received 22 May 1959

**Abstract:** Spectral representations for the amplitudes describing the decay of K-mesons and hyperons into two kinds of strongly interacting particles are derived. The singular integral equations for the amplitudes obtained in the "single-meson" approximation are solved by the method suggested by Muskhelishvili.

The work of Källén and Lehmann [1]) on the spectral representation of the one-particle Green's functions and, especially, Bogolubov's method of proving dispersion relations [2]) drew attention of many physicists to the problem of obtaining spectral representations for the vertex parts [3]).

The existence of such representations for complete vertices (and not merely for some vertex diagrams of perturbation theory) is far from self-evident and to prove their existence non-trivial mathematical methods are needed. Jost cites an example [4]) showing that it is impossible to obtain the spectral representation for the nucleon-meson vertex, on the basis of local commutation relations and the assumptions concerning the spectrum of masses. Similar difficulties arise in investigating the analytic properties of the nucleon-photon vertex (photon and meson being virtual).

It is especially interesting, therefore, to note that it is easy to justify dispersion relations for the vertices which correspond to the real processes of decay into two particles, provided the mass of the decaying particle is regarded as the energy variable. This fact has been noticed by Goldberger and Treiman [5]) who obtained and applied dispersion relations for the $(\pi \to \mu + \nu)$ decay.

As was said above, the present paper is concerned with spectral representations for the amplitudes describing the decay of K-meson and hyperon into two strongly interacting particles. To prove dispersion relations rigorously by the method suggested by Bogolubov [2]) is here possible only in a simplified manner since there is no unobservable region.

The dispersion relations in the one meson approximation are then solved by using the techniques developed by Muskhelishvili [9]), the amplitude of the process being determined within the accuracy of a constant real factor.

† On leave of absence from the Physical Institute of the Bulgarian Academy of Sciences.

The results thus obtained for arbitrary virtual energies of the decaying particle are analogous to those given by Takeda [6]).

In the recent work of Okubo, Marshak and Sudarshan [11]) † in which the decay of the $\Lambda$-hyperon is investigated on the basis of the theory of the axial-vector four-fermion interaction, dispersion relations for the $\Lambda$-hyperon decay are applied as well. The authors of the above paper examine an auxiliary problem assuming that the squares of all four momenta (of hyperon, nucleon and $\pi$-meson) equal the corresponding squares of masses, but the four-momentum conservation law does not apply, and that the $\Lambda$-hyperon is "totally uncoupled". In our work, on the contrary, the conservation law (1.8) holds and the square of the four-momenta of hyperon $q^2$ is regarded as a variable. The spin structure is therefore determined by two independent functions while in ref.[11]) four functions $\Omega_i$ are introduced, which rather complicates dispersion relations in the one meson approximation as well as their solutions. The final results are therefore different, though not contradictory ††.

## 1. Analytical Properties of Decay Amplitude. Derivation of Spectral Representations

We shall first consider the case when a hyperon decays into a nucleon and $\pi$-meson. At the close of this Section we shall mention the changes that have to be done to obtain dispersion relations for the decay of the K-meson into two $\pi$-mesons.

Let us denote the four-momenta of hyperon, nucleon and $\pi$-meson by $q$, $p$ and $k$ respectively. We shall assume that the four-momenta of decay products only obey the usual relation

$$p^2 = M^2, \qquad k^2 = \mu^2 \qquad (M \approx 6.8\mu) \tag{1.1}$$

while

$$q^2 = \omega^2 \tag{1.2}$$

will be considered as a variable (if all three masses are regarded as fixed, then, in virtue of the conservation law, there will be no invariant variable with respect to which the analytic properties of the decay amplitude might be considered). We put

$$J(x) = i\frac{\delta S}{\delta \psi(x)} S^\dagger, \qquad j(x) = i\frac{\delta S}{\delta \varphi(x)} S^\dagger, \tag{1.3}$$

where $\psi(x)$ and $\varphi(x)$ are the operators of the hyperon and meson fields respectively.

The matrix element of the process $Y \to N+\pi$ may be written in the form

$$\langle p, k|S|q\rangle = i\frac{(2\pi)^{\frac{1}{2}}}{\sqrt{2k_0}}\delta(p+k-q)T^{\mathrm{ret}}(k)U_Y(\mathbf{q}) \tag{1.4}$$

† The authors express their gratitude to S. Okubo for sending the preprint.

†† Cf. the footnote to (2.8).

where

$$T^{\mathrm{ret}}(k) = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \langle p|\frac{\delta J(x)}{\delta\varphi(x)}|0\rangle e^{ikx}d^4x = -\sqrt{2k_0}\langle p, k|J(0)|0\rangle \quad (1.5)$$

is the retarded amplitude, $d^4x = dx_0dx_1dx_2dx_3$,

$$kx = k_0x_0 - \mathbf{k}\cdot\mathbf{x}; \quad \delta(r) = \delta(r_0)\delta_3r_1)\delta(r_2)\delta\langle r_3).$$

We assume that the spinor $U_{\mathrm{Y}}(\mathbf{q})$ satisfies the Dirac equation describing particle with mass $\omega$ (1.2):

$$\gamma qU_{\mathrm{Y}}(\mathbf{q}) \equiv \begin{pmatrix} q_0 & -\boldsymbol{\sigma}\cdot\mathbf{q} \\ \boldsymbol{\sigma}\cdot\mathbf{q} & -q_0 \end{pmatrix} U_{\mathrm{Y}}(\mathbf{q}) = \omega U_{\mathrm{Y}}(\mathbf{q}). \quad (1.6)$$

The quantity hermitian conjugate to (1.5) and equal to the advanced amplitude may be written in the form

$$T^{\mathrm{adv}}(k) = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \langle p|\frac{\delta j(x)}{\delta\varphi(0)}|0\rangle e^{ikx}d^4x. \quad (1.7)$$

In virtue of the conservation law, following from (1.4),

$$q = p+k. \quad (1.8)$$

According to the Dirac equation the amplitude in ordinary space possesses a structure of the following type

$$T^{\mathrm{ret}}(k)U_{\mathrm{Y}}(\mathbf{q}) = \overline{U}_{\mathrm{N}}(\mathbf{p})\{\Omega^r{}_0+\Omega^r{}_1\gamma_5\}U_{\mathrm{Y}}(\mathbf{q}). \quad (1.9)$$

A similar structure may be written for the amplitude (1.7) (the indices in isobaric space are omitted). Here $\Omega_0$ and $\Omega_1$ are invariant functions; instead of $\omega$ it is convenient to choose as invariant variable

$$E \equiv \frac{1}{M}pk, \qquad (2ME = \omega^2 - M^2 - \mu^2). \quad (1.10)$$

The matrix $\gamma_5$ will be considered as Hermitian:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3i\gamma_0 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let us introduce a frame of reference in which the nucleon rests:

$$\begin{gathered} \mathbf{p} = 0, \\ p_0 = M, \quad k_0 = E, \quad \mathbf{k} = \mathbf{q} = \mathbf{e}\sqrt{E^2-\mu^2}, \quad |\mathbf{e}| = 1. \end{gathered} \quad (1.11)$$

To prove dispersion relations let us first consider a fictitious case when

$$k^2 = \tau < 0.$$

In this case the retarded amplitude (1.5) takes the form

$$T^{\mathrm{ret}}(E,\tau) = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \exp\{i(Ex_0 - \mathbf{e}\cdot\mathbf{x}\sqrt{E^2-\tau}\}\langle p|\frac{\delta J(0)}{\delta\varphi(x)}|0\rangle d^4x.$$

A similar expression is obtained also for $T^{\mathrm{adv}}$ from (1.7).

In consequence of the causality condition [2]) the amplitudes $T^{\mathrm{ret}}$ (or $T^{\mathrm{adv}}$) are analytic functions in the upper (or lower) semi-plane of the complex variable $E$. It will be shown that the difference between them vanishes along some interval of the real axis. For this consider the anti-Hermitian part of the amplitude

$$A(E,\tau) \equiv \frac{1}{2i}\{T^{\mathrm{ret}}(E,\tau)-T^{\mathrm{adv}}(E,\tau)\}. \tag{1.12}$$

Let us use the equality

$$\frac{\delta j(x)}{\delta\varphi(0)} - \frac{\delta J(0)}{\delta\varphi(x)} = i[j(x), J(0)] \tag{1.13}$$

and expand (1.13) over the complete system of the eigenfunctions of the energy-momentum operator

$$A(E,\tau) = \pi(2\pi)^{\frac{3}{2}} \sum_n \{\langle p|j(0)|p_n\rangle\langle p_n|J(0)|0\rangle\delta(M+E-\sqrt{M_n^2+E^2-\tau}) - \langle p|J(0)|p'_n\rangle\langle p'_n|j(0)|0\rangle\delta(E+\sqrt{M_n^2+E^2-\tau}). \tag{1.14}$$

Here $\sum_n$ implies integration over the continuous part of the spectrum and summation over the discrete characteristics of intermediate states. As is well known from the theory of spectral representations of Green's meson function [2]), the matrix element is

$$\langle p'_n|j(0)|0\rangle = 0$$

for $M_n'^2 < (3\mu)^2$ in virtue of stability of one-particle states and invariance with respect to space reflections. Since $p'_{n_0}$ is positive we conclude that the second term of the right hand side in (1.14) vanishes when

$$\tau < (3\mu)^2. \tag{1.15}$$

The singularities of the first $\delta$-function of eq. (1.14) lie at the points

$$E_n = \frac{1}{2M}(M_n^2 - M^2 - \tau). \tag{1.16}$$

It can be seen from (1.14) that, generally, there exists one discrete pole of the "one-nucleon" term for $M_1 = M$:

$$E = -E_1(\tau) = -\frac{\tau}{2M}. \tag{1.17}$$

The residue at this pole is proportional to the product of the strong and weak coupling constants and has the same spin structure as the amplitude (1.9).

The continuous spectrum for the first term in (1.14) starts with the value $M_n = M+\mu$:

$$E_n \geqq \frac{2M\mu+\mu^2-\tau}{2M} \equiv E_c(\tau). \tag{1.18}$$

Hence it is clear that the functions $(E+E_1)T^{\mathrm{ret}}(E,\tau)$ and $(E+E_1)T^{\mathrm{adv}}(E,\tau)$ coincide for $\tau < (3\mu)^2$:

$$-\infty < E < E_c(\tau). \tag{1.19}$$

According to Bogolubov's theorem [2]), therefore (cf. also ref.[7]) where more precise theorems are adduced), there exists a single function $T(E,\tau)$ analytic with respect to $E$ and $\tau$ in the region

$$|\mathrm{Im}\sqrt{(E^2-\tau)}| < |\mathrm{Im}\sqrt{(E+3\mu)(E-\mu)}| \tag{1.20}$$

and coinciding with $T^{\mathrm{ret}}$ and $T^{\mathrm{adv}}$ for the real $E$ and $\tau$ which satisfy (1.19). It is easy to see that the inequality (1.20) holds for $\mathrm{Im}\, E \neq 0$ or $E = \mathrm{Re}\, E < \mu$ for $\tau = \mu^2$. Consequently, in virtue of (1.15), the function can be continued analytically with respect to $\tau$ up to $\tau = \mu^2$. For the real case of interest to us $\tau = \mu^2$, the boundary of the continuous spectrum

$$E_c(\mu^2) = \mu$$

coincides with the threshold of the reaction under consideration. As is known[2]), the functions $T(E)$ do not increase at infinity faster than a polynomial. For the sake of simplicity we suppose that $T(E)$ is bounded at infinity. Employing the Cauchy integral theorem for the function $T(E)/E$ and performing transition to the limit $\mathrm{Im}\, E \to 0$, we obtain the following dispersion relations:

$$\mathrm{Re}\, T(E) = \frac{E}{\pi}\mathrm{P}\int_0^\infty \frac{\mathrm{Im}\, T(E')}{E'(E'-E)}\,\mathrm{d}E' + \frac{c}{\pi}\,\frac{E}{E_1}\,\frac{1}{E+E_1} + T(0).$$

According to (1.9), and since the anti-Hermitian part $A(E,\tau)$ has the same structure as the amplitude

$$A(E_1\tau)U_Y(\mathbf{q}) = \pi(2\pi)^{\frac{3}{2}}\frac{M+E}{M}\sum_n \langle p|j(0)|p_n\rangle\langle p_n|J(0)|0\rangle\delta(E-E_n(\tau))U_Y(\mathbf{q})$$
$$= \bar{U}_N(\mathbf{p})(A_0(E)+A_1(E)\gamma_5)U_Y(\mathbf{q}), \tag{1.21}$$

it can easily be seen that the invariant functions $\Omega_l(E)$, with $l = 0, 1$, obey the same dispersion relations. They can be written in the form

$$\Omega_l^r(E) = \frac{E}{\pi}\int_\mu^\infty \frac{\mathrm{Im}\,\Omega_l^r(E')}{E'(E'-E-i\varepsilon)}\,\mathrm{d}E' + \frac{E}{E_1}\,\frac{a_l}{E+E_1+i\varepsilon} + \Omega_l(0) \tag{1.22}$$

where $a_l$ and $\Omega_l(0)$ are real constant quantities.

The relations (1.22) and (1.21) yield spectral representations for the invariant functions $\Omega_l(E)$, the variable $E$ being also determined in an invariant manner (1.10).

The process $(K \to 2\pi)$ is much simpler since the amplitude has no spin structure. The use of the stability of states with one $\pi$-meson yields an error of second-order smallness with respect to the weak interaction constant. In the

case of ($K \to 2\pi$) decay no discrete pole appears in the imaginary part of the imaginary amplitude because of pseudoscalarity.

In this case the dispersion relations assume the form

$$T^{\mathrm{ret}}(E) = \frac{E}{\pi}\int_{\mu}^{\infty} \frac{\operatorname{Im} T^{\mathrm{ret}}(E')}{E'(E'-E-i\varepsilon)}\,\mathrm{d}E' + T(0), \tag{1.23}$$

where

$$\operatorname{Im} T^{\mathrm{ret}}(E) = \pi(2\pi)^{\frac{3}{2}}\,\frac{\mu+E}{\mu}\sum_{\substack{n\\ M_n \geq 2\mu}} \langle p|j(0)|p_n\rangle\langle p_n|J_{\mathrm{K}}(0)|0\rangle\,\delta\left(E-\frac{M_n^2-2\mu^2}{2\mu}\right) \tag{1.24}$$

is the K-meson current $J_{\mathrm{K}}$ determined just as the current $J$, eq. (1.3).

## 2. Approximate Solution of Dispersion Relations

### 2.1. DECAY OF K-MESONS

Our application of the dispersion relations here obtained is confined to the "one-meson state" in the expansions (1.21), (1.24) for the anti-Hermitian parts of the relevant amplitudes.

Let us start with the decay of K-mesons. In this case the "one-meson" approximation corresponds to an intermediate state with two $\pi$-mesons. For the sake of definiteness we confine ourselves to the investigation of the $K^0$-meson decay. The decay of a charged K-meson into two $\pi$-mesons bends itself to a simpler treatment since this reaction is at work in one channel only.

As is well known [8]), from CP-invariance it follows that only the $K_1^0$-meson decays into two $\pi$-mesons where

$$|K_1^0\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right).$$

We write dispersion relations for the amplitude which corresponds to a certain value of the total isotopic spin of the system of two $\pi$-mesons. In the decay of $K^0$-meson $I$ equals 0 and 2; the amplitudes $T_I$ for the decay into particles with definite charge are related through the Clebsch-Gordan coefficients

$$\begin{aligned} T_0 &= \sqrt{\tfrac{2}{3}}\,T_{+-} - \frac{1}{\sqrt{3}}\,T_{00}, \\ T_2 &= \frac{1}{\sqrt{3}}\,T_{+-} + \sqrt{\tfrac{2}{3}}\,T_{00}, \end{aligned} \tag{2.1}$$

where

$$T_{+-} \equiv T(\pi^+, \pi^-|K_1^0); \qquad T_{00} \equiv T(\pi^0, \pi^0|K_1^0).$$

If we confine ourselves to intermediate states with two $\pi$-mesons in the expansion (1.24) for the imaginary part of the amplitude, we shall obtain in virtue of the isobaric spin invariance of scattering of decay products

$$\operatorname{Im} T_I^{\mathrm{ret}}(E) = \mathrm{e}^{-i\delta_I(E)} \sin \delta_I(E) T_I^{\mathrm{ret}}(E) \tag{2.2}$$

where $\delta_I$ is a phase shift for the s-wave with the isobaric spin $I$ of the $\pi$—$\pi$ scattering amplitude (since K-meson is scalar only s-waves contribute in the scattering amplitude of the decay products).

From (2.2) one obtains

$$T_I^{\mathrm{ret}}(E) = f_I(E)\mathrm{e}^{i\delta_I(E)} \tag{2.3}$$

where $f_I(E)$ is a real function of $E$. To determine this function we may use the dispersion relation. Employing (2.2), the dispersion relation (1.23) can be represented as follows

$$T_I^{\mathrm{ret}}(E) = \frac{E}{\pi}\int_{\mu}^{\infty} \frac{h_I(E')T_I^{\mathrm{ret}}(E')}{E'(E'-E-i\varepsilon)}\,\mathrm{d}E' + T_I(0), \tag{2.4}$$

the notation here being

$$h_I(x) = \mathrm{e}^{-i\delta_I(X)} \sin \delta_I(x).$$

The singular integral equation is easily solved by Muskhelishvili's method[9]). The reader may also confer the paper of Omnes where the author specially examines the equations of the type (2.4) but without the factor $E/E'$ under the integral sign. Consider the function $T_I(z)$ of the complex variable $z$:

$$T_I(z) = \frac{z}{\pi}\int_{\mu}^{\infty} \frac{h_I(E')T_I^{\mathrm{ret}}(E')}{E'(E'-z)}\,\mathrm{d}E'.$$

From the definition of $T_I(z)$ follow the relations (for real values of $E$)

$$T_I(E_+) = T_I^{\mathrm{ret}}(E)$$

$$T_I(E_+) - T_I(E_-) = (1-\mathrm{e}^{-2i\delta_I(E)})T_I^{\mathrm{ret}}(E)\theta(E-\mu)$$

where

$$E_\pm = E \pm i\varepsilon, \qquad \varepsilon > 0, \qquad \theta(x) = \begin{cases} 1, & x > 0 \\ 0 & x < 0. \end{cases}$$

The function $T_I(z)$ is therefore the solution of Hilbert's boundary problem

$$T_I(E_+)\mathrm{e}^{-2i\delta_I(E)\theta(E-\mu)} - T_I(E_-) = 0.$$

Under our assumption that $T_I(z)$ does not increase at infinity the solution of this problem has the form (for real values of $E$)

$$T_I^{\mathrm{ret}}(E) = C_I \exp\left\{\frac{E}{\pi}\mathrm{P}\int_{\mu}^{\infty} \frac{\delta_I(x)}{x(x-E)}\,\mathrm{d}x + i\delta_I(E)\theta(E-\mu)\right\} \tag{2.5}$$

where

$$C_I = T_I(0)$$

is a real constant which may be determined (if the phase shifts of the $\pi$—$\pi$ scattering are assumed to be known) by the experimental values of the decay time or theoretically (following the perturbation theory), which involves a

hypothesis about the concrete form of the interaction responsible for the decay under consideration.

To go over to measurable physical quantities we make use of the relations (2.1). For the real decay

$$\omega = \sqrt{q^2} = m_K \approx 494 \text{ MeV},$$

$$E = E_0 \equiv \frac{m_K{}^2 - 2\mu^2}{2\mu} \approx 5.5\mu \approx 750 \text{ MeV},$$

which corresponds to the total kinetic energy in the centre-of-mass system

$$W_K = m_K - 2\mu \approx 220 \text{ MeV}.$$

According to (2.1) and (2.3) we obtain for the probabilities of decay into $\pi^+ + \pi^-$ and $\pi^0 + \pi^0$

$$\begin{aligned} |T_{+-}|^2 &= \tfrac{1}{3} f_2^2(E_0) + \tfrac{2}{3} f_0^2(E_0) + \frac{2\sqrt{2}}{3} f_0(E_0) f_2(E_0) \cos\left(\delta_2(E_0) - \delta_0(E_0)\right), \\ |T_{00}|^2 &= \tfrac{2}{3} f_2^2(E_0) + \tfrac{1}{3} f_0^2(E_0) - \frac{2\sqrt{2}}{3} f_0(E_0) f_2(E_0) \cos\left(\delta_2(E_0) - \delta_0(E_0)\right). \end{aligned} \tag{2.6}$$

Introduce the notation

$$\lambda = \frac{f_2(E_0)}{f_0(E_0)}.$$

For the ratio of these probabilities we obtain the well-known result [6])

$$\frac{W(K_1^0 \to \pi^0 + \pi^0)}{W(K_1^0 \to \pi^+ + \pi^-)} = \frac{2\lambda^2 + 1 - 2\sqrt{2}\,\lambda \cos(\delta_2 - \delta_0)}{\lambda^2 + 2 + 2\sqrt{2}\,\lambda \cos(\delta_2 - \delta_0)}.$$

It is evident that this ratio equals $\frac{1}{2}$ not only for $\lambda = 0$ (which corresponds to the selection rule with respect to the isobaric spin $|\Delta I| = \frac{1}{2}$) but for one more definite value $\lambda$ depending on $\delta_2 - \delta_0$.

### 2.2. DECAY OF HYPERONS

The solution of dispersion relations for hyperon decay is carried out in a similar manner except for some insignificant complication resulting from the spin structure and "one-nucleon" pole of the amplitude. Therefore, it is not repeated in detail.

Since parity is conserved in $\pi$-meson-nucleon scattering, the orbital moment $l = j \pm \frac{1}{2}$ is likewise conserved in scattering of hyperon-decay products as well as the total angular momentum $j$ of the system. It is not difficult to show that the invariant functions $\Omega_0$ and $\Omega_1$ introduced in (1.9) are eigenfunctions of the orbital moment operator and correspond to the eigenvalues $l = 0$ and $l = 1$. Suffice it to note that in the centre-of-mass system ($\mathbf{q} = \mathbf{p} + \mathbf{k} = 0$) the decay amplitude has the form

$$T_{s's}(E) U_Y(\mathbf{q}) = \sqrt{\frac{M + p_0}{2M}} \left\{ \Omega_0(E) \delta_{s's} - \Omega_1(E) \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})_{s's}}{M + p_0} \right\} \tag{2.7}$$

where

$$p_0 = \frac{\omega^2+M^2-\mu^2}{2\omega} = M\frac{M+E}{\sqrt{M^2+\mu^2+2ME}}$$

is the energy of the nucleon in the centre-of-mass system.

We rewrite dispersion relations for the amplitudes of the state with definite isobaric spin of the system of decay products $\Omega_{Il}(l = 0,\ 1,\ I = \frac{1}{2},\ \frac{3}{2})$. After performing a phase shift analysis for the meson-nucleon scattering amplitude we obtain † in the expression for the imaginary part (1.21)

$$\operatorname{Im}\Omega^r_{Il}(E) = \pi a_{Il}\delta(E+E_1)+h_{Il}\Omega^r_{Il}\theta(E-\mu) \tag{2.8}$$

where

$$E_1 = \frac{\mu^2}{2M},\ h_{Il} = e^{-i\delta_{Il}(E)}\sin\delta_{Il}(E) \tag{2.9}$$

($a_{Il}$ are real constant quantities, $\delta_{Il}$ are phase shifts of meson-nucleon scattering). From (2.8) it follows that for $E > \mu$,

$$\Omega^r_{Il}(E) = f_{Il}(E)e^{i\delta_{Il}(E)}, \tag{2.10}$$

$f_{Il}(E)$ being a real function. To determine the latter we make use again of the dispersion relations which can be written in the form

$$\Omega^r_{Il}(E) = \frac{E}{E_1}\frac{a_{Il}}{E+E_1+i\varepsilon}+\frac{E}{\pi}\int_\mu^\infty \frac{h_{Il}(E')\Omega^r_{Il}(E')}{E'(E'-E-i\varepsilon)}\,dE'+\Omega_{Il}(0). \tag{2.11}$$

This equation yields a solution which does not increase at infinity and has the form for the complex argument [3])

$$\Omega_{Il}(z) = \left\{\frac{z}{E_1}\frac{a_{Il}}{z+E_1}e^{-\frac{E_1}{\pi}\int_\mu^\infty \frac{\delta_{Il}(x)}{x(x+E_1)}\,dx}+\Omega_{Il}(0)\right\}\exp\left\{\frac{z}{\pi}\int_\mu^\infty \frac{\delta_{Il}(x)}{x(x-z)}\,dx\right\}.$$

For real $E > \mu$ it assumes the following form:

$$\Omega^r_{Il}(E) = \Omega_{Il}(E+i\varepsilon)$$

$$= \left\{\frac{E}{E_1}\frac{a_{Il}}{E+E_1}e^{-\frac{E_1}{\pi}\int_\mu^\infty \frac{\delta_{Il}(x)}{x(x+E_1)}\,dx}+\Omega_{Il}(0)\right\}\exp\left\{\frac{E}{\pi}P\int_\mu^\infty \frac{\delta_{Il}(x)}{x(x-E)}\,dx+i\delta_{Il}(E)\right\}. \tag{2.12}$$

It is interesting to apply these results to the decay of $\Sigma^+$ and that of $\Lambda^0$ hyperons which take place in two channels with respect to the charge. For this it is necessary to change over from the amplitudes with definite isotopic spin to those of the decay into definite particles through the Clebsch-Gordan coefficients, as in ref.[8]), for example, and to assume $\omega$

† The relations (2.8), (2.9) are obtained easily from eqs. (2.5), (2.6) in the work of Okubo *et al.*[11]) under the assumption that the spinor $U_\nu(\mathbf{q})$ satisfies the Dirac equation with a mass $\omega$. Between the amplitudes $M_i^I$ of the above-mentioned paper and our amplitudes $\Omega_{Il}$ there exists simple connection

$$\Omega_{I0} = M_1^I-(\omega-M)M_3^I,\qquad \Omega_{I1} = M_2^I-(\omega-M)M_4^I.$$

As was already pointed out, in ref.[11]) it is assumed that $U_\nu(\mathbf{q})$ obeys the Dirac equation with real hyperon mass. Therefore, here and later the authors [11]) obtain considerably more complicated relations.

to be equal to the real mass of the corresponding hyperon. The total kinetic energy which releases in the centre-of-mass system being $m_K \approx 38$ MeV, this yields

$$E = \frac{m_\Lambda{}^2 - M^2 - \mu^2}{2M} \approx 183 \text{ MeV}$$

in the case of decaying $\Sigma$-hyperon, and

$$E = \frac{m_\Sigma{}^2 - M^2 - \mu^2}{2M} \approx 275 \text{ MeV}$$

in the case of decaying $\Sigma^+$ hyperon.

It appears to be worthwhile to compare the formulas thus obtained with experimental data on decay and meson-nucleon scattering, after the constants in (2.12) have been determined by using the perturbation theory and proceeding from a concrete type of interaction (cf.[11])).

The authors are deeply grateful to A. A. Logunov who guided this work. They also thank P. S. Isaev, S. M. Bilenky and R. M. Ryndin for valuable comments and discussions.

*Note added in proof*: The authors are indebted to M. G. Krein who drew their attention to the fact that a strict solution of the Riemann-Hilbert problem, on which the method of solving singular integral equations is based, was first put forth by F. D. Gakhov [12]).

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# ON A POSSIBLE TEST OF QUANTUM ELECTRODYNAMICS IN ELECTRON SCATTERING

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Received 27 April 1959

**Abstract:** It is shown that a violation of quantum electrodynamics of the type corresponding to an effective nonlocal structure of an electron may be found through a study of the elastic scattering of electrons on α-particles (with a very small improvement in accuracy of measurements as compared with the level attained so far). Because of the absence of spin and magnetic moment elastic scattering on α-particles is generally a very convenient process for detecting any kind of violation of the theory in the region of not very small distances (of the order of $10^{-13}$ cm to $10^{-14}$ cm).

Unlike the case of α-particles, the scattering of electrons on nucleons offers in principle no possibility of detecting violations of the theory of the type under consideration (if one excludes such complex experiments as measuring changes in the polarization of electrons in scattering or the scattering of polarized electrons on polarized nucleons).

## 1. Introduction

The problem under discussion is this: is it possible to obtain any information on the structure of an electron, i.e., to find any deviations from the concepts of quantum electrodynamics on the basis of experimental data on the scattering of high energy electrons on α-particles and nucleons?

We assume that the electron is non-punctiform and that its interaction with the electromagnetic field is described by the vertex part $\Gamma_\mu^{(e)}$ of the most general (vector) type

$$\Gamma_\mu^{(e)} = \gamma_\mu a_e(q^2) + i\gamma_\mu\gamma_\nu \frac{q_\nu}{\sqrt{q^2}} b_e(q^2) \tag{1}$$

where $q$ is the four-momentum transfer in scattering ($q^2 = q^2 - q_0^2$); the real functions $a_e(q^2)$ and $b_e(q^2)$ correspond to the form-factors of the charge and anomal magnetic moment of an electron. The case of the point electron is characterised by the values $a_e \equiv 1$, $b_e \equiv 0$ (or more accurately by values different from these by quantities of the order of $\frac{1}{137} = e^2$); if the electron is non-punctiform, then $a_e$ and $b_e$ may differ considerably from these values, and in general their ratio

$$\mu_e(q^2) = b_e(q^2)/a_e(q^2) \tag{2}$$

must have a value of the order of unity. We do not think that a deviation † of the future theory from the current one of the type (1) with $a_e \neq 1$ and $\mu_e \neq 0$ is the most probable one. Nevertheless, the problem is so important that an experimental check of the presence or absence of the deviation under consideration is doubtless of great interest.

It will be shown that if this violation of the theory takes place at distances in the range of $r = 1/q \approx 10^{-13}$ to $10^{-14}$ cm (and even down to $10^{-15}$ cm), it may easily be detected by scattering of electrons on α-particles.

## 2. Elastic Scattering of Electrons on α-Particles

In calculating the scattering cross-section we use the most general form of the vertex part $\Gamma_\mu^{(\alpha)}$ for the interaction of α-particles with the electromagnetic field

$$\Gamma_\mu^{(\alpha)} = 2i(P_{1\mu}+P_{2\mu})a_\alpha(q_2) \tag{3}$$

where $a_\alpha(q^2)$ is the form-factor of the charge distribution in the α-particle (for $q^2 = 0$, one has $a_\alpha = 1$). From (1), (2), (3) and taking into account the

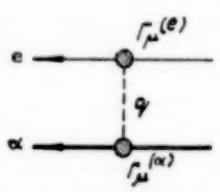


Fig. 1. Feynman graph for scattering the electron (e) on α-particle (α). $\Gamma_\mu^{(e)}$ and $\Gamma_\mu^{(\alpha)}$ are full vertex operators of the electron and α-particle.

simplest graph of fig. 1, we obtain for the differential scattering cross-section $d\sigma/d\Omega = \sigma(\vartheta, \varepsilon)$ (in the laboratory system)

$$\sigma = \sigma_0\left(A + \frac{B}{\cos^2\frac{1}{2}\vartheta}\right) \tag{4}$$

where

$$\sigma_0(\vartheta, \varepsilon) = \left(\frac{Ze^2}{2\varepsilon}\right)^2 \frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta\left(1+\frac{2\varepsilon}{M}\sin^2\frac{1}{2}\vartheta\right)} \tag{5}$$

is the scattering cross-section of a point electron on a point α-particle, $Z = 2$ is the charge, $M$ the mass of α-particle, while the quantities $A$ and $B$ depend only on $q^2$:

† Pomeranchuk (particular communication) noticed that this deviation is probably incompatible with the dispersion relations for scattering of photons on electron if the coupling is weak, i.e., if $a_e$ and $b_e$ do not become very large.

$$A = \left[1 - \frac{\Delta}{2M}\mu_e^2(q^2)\right] F(q^2), \qquad B = \left[1 + \frac{\Delta}{2M}\right] \mu_e^2(q^2) F(q^2). \tag{6}$$

Here † $F(q^2) = a_e^2(q^2)a_\alpha^2(q^2)$, and $\Delta = q^2/2M$ is the recoil kinetic energy of $\alpha$-particle. The angle $\vartheta$ of scattering is connected with the energy of electrons and the quantity $\Delta$ (or with $q^2$) by the relation

$$\sin^2 \tfrac{1}{2}\vartheta = \frac{M\Delta}{2\varepsilon(\varepsilon - \Delta)} \equiv \frac{q^2}{4\varepsilon(\varepsilon - q^2/2M)}; \tag{7}$$

small quantities of the order $m_e/\varepsilon$, or $m_e/q$, where $m_e$ is the mass of an electron, being neglected in all formulae.

For a point electron $\mu_e = 0$, i.e., according to (6), $B = 0$ as well. From (4) it follows that in this case the value of the experimentally measured ratio

$$\lambda = \frac{\sigma}{\sigma_0} \tag{8}$$

should not depend on energy (or on scattering angle $\vartheta$) for fixed $q^2$.

If for any $q^2$ such dependence is observed, it will testify to the violation of present theory. Even more conclusively, if this dependence proves to be of the special kind (4) $A + (B/\cos^2 \frac{1}{2}\vartheta)$, this will denote that the violation of the theory is of the type considered. This being the case, the experimental data on the quantity (8) for two values $\varepsilon_1$ and $\varepsilon_2$ of energy of electrons, but for the same $q^2$, will make it possible to find the values $A$ and $B$, i.e. to calculate the quantity $\mu_e(q^2)$ from the relation

$$\frac{A}{B} = 1 - \frac{\Delta}{2M + \Delta}(1 + \mu_e^2).$$

To illustrate the scale of the dependence of the relation (8) on energy (for $q^2 = \text{const.}$) which may be expected if $\mu_e \neq 0$, the relevant graphs are given in the figs. 2, 3 and 4 for certain values of the quantities $\mu_e^2$ and $q = \sqrt{2M\Delta}$. (The curves are normalized to unity at $\varepsilon \to \infty$, i.e., the value

$$(\lambda(q, \varepsilon)/\lambda(q, \infty) = (1 + \mu_e^2)^{-1} + \frac{\mu_e^2}{(1 + \mu_e^2)\cos^2 \frac{1}{2}\vartheta}$$

is marked off on the axis of the ordinates, $\cos^2 \frac{1}{2}\vartheta$ being determined by the relation (7)). It may be seen, for example, that for $q = 300$ MeV and if $\mu_e^2 = 0.40$, the quantity $\lambda$ increases approximately by fifty per cent if the energy $\varepsilon = 400$ MeV changes over to $\varepsilon = 200$ MeV. Such a large variation of $\lambda$ is evidently excluded by the experimental data already available.

† Alongside the change (1) of the vertex part, violating quantum electrodynamics, deviations in Green's functions may likewise be considered; as a result the function $F(q^2)$ will also contain the factor $d^2(q^2) = [q^2 D(q^2)]^2$ (where $D$ is the photon Green function), which is quite inessential for the following.

The scattering of electrons on gaseous helium was studied by McAllister and Hofstadter [1]) for electron energies (in the laboratory system) of 188 MeV,

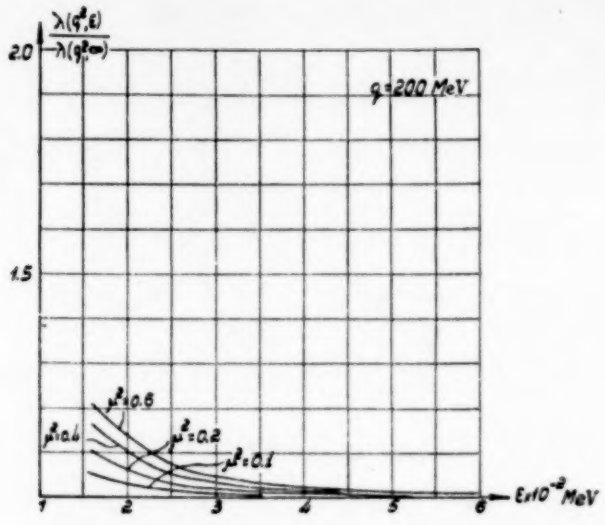


Fig. 2. Dependence of the scattering cross-section $\sigma$ of electrons on $\alpha$-particles on the energy of the electron $\varepsilon$ for four values of $\mu_e{}^2$ : 0.1; 0,2; 0.4 and 0.6, at the fixed value $q = 200$ MeV.

The quantity $\lambda(q^2, \varepsilon)/\lambda(q^2, \infty)$ is marked off on the axis of the ordinates where $\lambda = \sigma/\sigma_0$. If the values $1/\cos^2 \frac{1}{2}\vartheta$ were marked off on the axis of the abscisses, instead of $\varepsilon$, a set of straight lines would result.

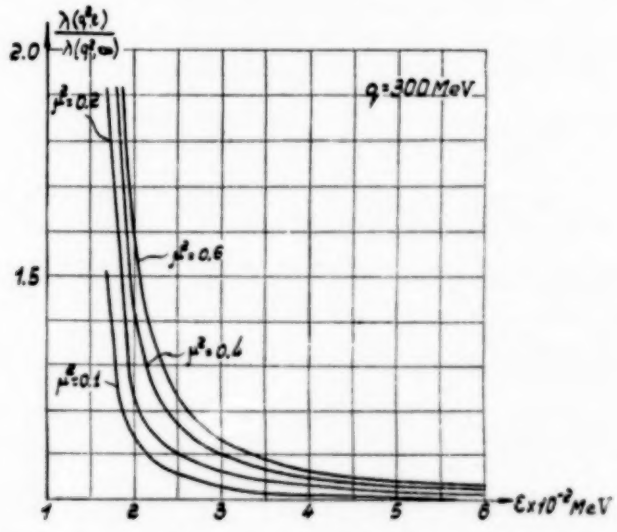


Fig. 3. The same as in fig. 2 for $q = 300$ MeV.

and by Blankenbecler and Hofstadter [2]) for energies of 400 MeV. The values of the ratio (8) obtained by them are given in fig. 5 (on a logarithmic scale); the quantity $q = \sqrt{2M\Delta}$ of the transferred momentum † is marked off on the axis of the abscissae of this graph. The circles correspond to $\varepsilon = 180$ MeV,

† To gain an idea of the range of distances it should be noted that the value $q \approx 200$ MeV corresponds to a distance $r = 1/q$ of the order of $10^{-13}$ cm.

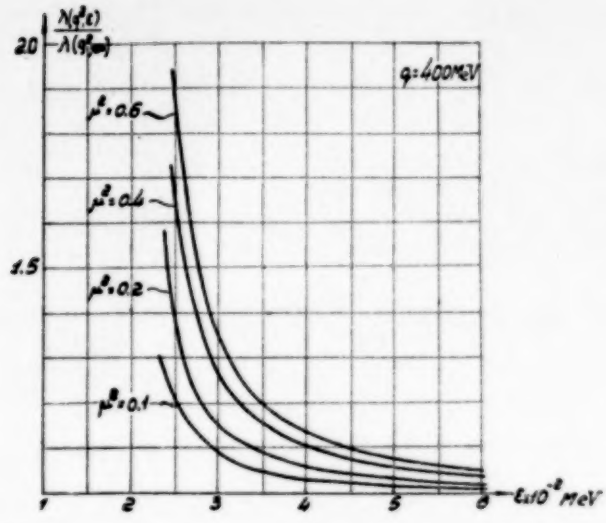


Fig. 4. The same as in fig. 2 for $q = 400$ MeV.

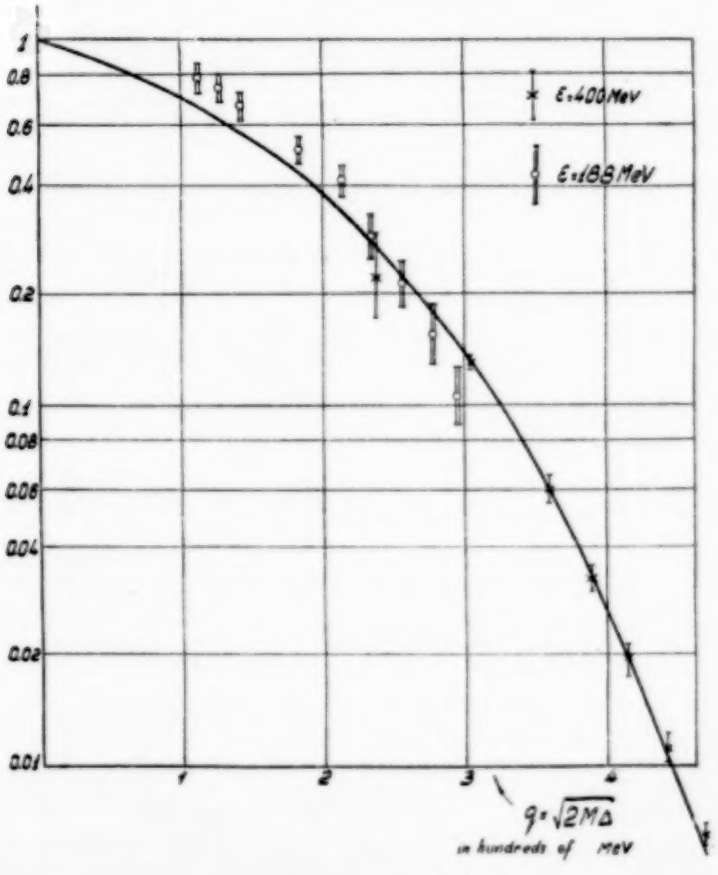


Fig. 5. Experimental dependence of the cross-section of scattering of an electron on gaseous helium on the four-momentum transfer $q = \sqrt{2M\Delta}$. The value $\lambda = \sigma/\sigma_0$ is marked off on the axis of the ordinates $0 - \varepsilon = 188$ MeV; $\times - \varepsilon = 400$ MeV. If eq. (4) is valid, the points for 188 MeV should lie, for each value of $q$, above the points for 400 MeV. It may be seen that a 50 per cent increase in $\lambda = \sigma/\sigma_0$ as $\varepsilon = 400$ MeV changes over to $\varepsilon = 188$ MeV for $q = 300$ MeV is almost certainly excluded in spite of large experimental errors.

the crosses to $\varepsilon = 400$ MeV, the graph in fig. 5 is drawn as smoothly as possible through the point $\varepsilon = 400$ MeV. If the formula (4) is true, the points designated by circles ($\varepsilon = 180$ MeV) for each $q^2$ should lie above the points designated by crosses ($\varepsilon = 400$ MeV), i.e., above the curve. This actually takes place only for the left part of the graph, but elsewhere the shift of the one group of points as compared with the other lies within the limits of experimental errors. From the data of fig. 5 it is clear that a 50 per cent increase in the value of the rate $\lambda = \sigma/\sigma_0$ for $q = 300$ MeV when $\varepsilon = 400$ MeV changes over to $\varepsilon = 180$ MeV is almost certainly excluded, even allowing for the considerable experimental inaccuracies. It is likewise clear that if the accuracy of measurement were increased at least several times (and if more than two measurements were carried out for the given $q^2$), much more definite inferences may be drawn.

If it turns out that the quantity $\lambda = \sigma/\sigma_0$ depends only weakly on the energy $\varepsilon$, this may be interpreted in two ways: either $\mu_e$ is small but differs from zero, or $\mu_e \equiv 0$ (point electron) and the weak dependence of $\lambda$ on energy is due to the contribution of radiation corrections.

In this case the value of the ratio $\lambda = \sigma/\sigma_0$ obtained experimentally should, first of all, be compared with a theoretical value corresponding to a point electron with due account of radiation corrections [3,4,5]):

$$\frac{d\sigma}{d\Omega\sigma_0} = a_\alpha{}^2(q^2)(1+\delta_B-\delta_R), \tag{9}$$

where

$$\delta_B = 2\pi e^2 \frac{\sin\frac{1}{2}\vartheta}{1+\sin\frac{1}{2}\vartheta},$$

$$\delta_R = \frac{e^2}{\pi}\left\{4\ln\frac{\varepsilon}{\Delta\varepsilon}\ln\frac{2\varepsilon\sin\frac{1}{2}\vartheta}{m} - \frac{13}{3}\ln\frac{2\varepsilon\sin\frac{1}{2}\vartheta}{m} - 2\ln\frac{2\varepsilon}{m} + \frac{28}{9} + 2\sin^2\tfrac{1}{2}\vartheta f(\vartheta)\right\},$$

$$f(\vartheta) = \frac{1}{\sin\frac{1}{2}\vartheta}\int_{\cos\frac{1}{2}\vartheta}^{1}\left[\frac{\ln\frac{1+\xi}{2}}{1-\xi} - \frac{\ln\frac{1-\xi}{2}}{1+\xi}\right]\frac{d\xi}{\sqrt{\xi^2-\cos^2\frac{1}{2}\vartheta}}, \qquad \frac{\varepsilon}{m}\sin\tfrac{1}{2}\vartheta \gg 1.$$

Here $\Delta\varepsilon \ll \varepsilon$ is a minor loss of energy in non-elastic scattering; $\sigma_0$ is determined by eq. (5). Eq. (9) is written without considering the effect of the recoil of the $\alpha$-particle, i.e., neglecting terms of the order $\varepsilon/M$ which are small if energies are not too high. If the experimental dependence of $d\sigma/d\Omega\sigma_0$ on the energy $\varepsilon$ has the concrete form (9) shown in fig. 6, it is clear that we have a point electron. Otherwise, if the contribution from the terms with $\mu_e \neq 0$ in the energy dependence of the ratio $d\sigma/d\Omega\sigma_0$ is essential, the electron is a non-punctiform particle.

## 3. Scattering of Electrons on Nucleons

It may turn out that at distances of the order of $10^{-13}$ to $10^{-14}$ cm (i.e. up to the values of $q$ in the GeV range) no violation of the theory, i.e., no dependence of the ratio (8) on energy, will be observed. As the values

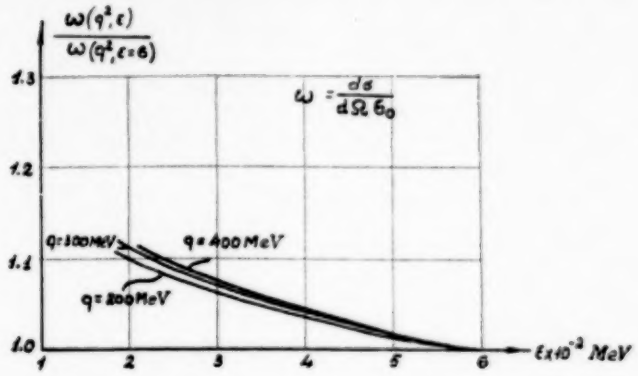


Fig. 6. Dependence on energy of the cross-section of scattering of point electrons on α-particles with account for radiation corrections for three fixed values of $q$. The value $\omega(q^2, \varepsilon)/\omega(q^2, \varepsilon = \sigma)$, where $\omega = \mathrm{d}\sigma/\mathrm{d}\Omega\sigma_0$, are marked off on the axis of the ordinates. Comparison with the graphs of figs. 3 and 4 shows that the contribution due to the non-punctiform character of the electron may be distinguished from that of radiation corrections with point electrons.

$q$ become larger, the cross-section of scattering on α-particles diminishes rapidly owing to the relatively large radius of the charge distribution in the α-particle (i.e. to a rapid decrease of $a_\alpha(q^2)$), and to the competition from non-elastic processes. A question arises: what may be gained from the scattering of electrons on nucleons (which is more easily performed than scattering on α-particles in the region of very large $q$).

The vertex part corresponding to the interaction of a nucleon with the electromagnetic field has the following general form [6,7,8]):

$$\Gamma_\mu^{(\mathrm{N})} = \gamma_\mu a_\mathrm{N}(q^2) + i\gamma_\mu\gamma_\nu \frac{q_\nu}{2M} b_\mathrm{N}(q^2), \tag{10}$$

where $a_\mathrm{N}$ and $b_\mathrm{N}$ are the form-factors of the nucleon charge and anomalous magnetic moment; $M$ now designates the mass of a nucleon.

Using eq. (1) and (10) for vertex parts and taking into account the simplest graph of fig. 1 we obtain, after some computation, an expression for the differential scattering cross-section of electrons on nucleons (in the laboratory system) which is quite similar to (4) and (5), but in which $Z = 1$ while $A$ and $B$ have the values

$$\begin{aligned} A &= \left[1 - \frac{\Delta}{2M}\mu_\mathrm{e}^2\right] F_\mathrm{N} - \left[1 - \left(1 - \frac{\Delta}{2M}\right)\mu_\mathrm{e}^2\right] \Phi_\mathrm{N}, \\ B &= \left(1 + \frac{\Delta}{2M}\right)\mu_\mathrm{e}^2 F_\mathrm{N} - \left(2 + \frac{\Delta}{2M}\mu_\mathrm{e}^2\right)\Phi_\mathrm{N}, \end{aligned} \tag{11}$$

where

$$\Phi_N = \frac{\Delta}{2M}(a_N+b_N)^2 a_e^2, \qquad F_N = \left(a_N^2+\frac{\Delta}{2M}b_N^2\right)a_e^2-\Phi_N;$$

in the region $q \approx \Delta \approx M \approx 1$ GeV, both $\Phi_N$ and $F_N$ have values of the same order.

According to (11), for a point electron ($\mu_e \equiv 0$) the value of $B$ does not vanish, unlike the case of $\alpha$-particle (which is evidently connected with the nucleon having spin and magnetic moment). In the case of scattering on nucleons, therefore, the rate $\sigma/\sigma_0$ for $q^2 = \text{const.}$ depends on the energy $\varepsilon$ even for a point electron.

If experiment shows that this dependence is not of the special kind (4), (7):

$$A+\frac{B}{\cos^2\frac{1}{2}\vartheta} = A+B\Big/\left(1-\frac{\Delta M}{2\varepsilon(\varepsilon-\Delta)}\right),$$

the conclusion that the current theory is violated is self-evident. (This has been noted by Akhiezer, Rosenzveig and Shmushkevitch [6]); in the case of a point electron, $\mu_e \equiv 0$, $a_e \equiv 1$, formulae (4), (5) and (11) coincide with eq. (28) of their paper for the scattering cross-section.) If on the other hand, the dependence of the ratio $\sigma/\sigma_0$ on the energy $\varepsilon$ for $q^2 = \text{const.}$ conforms to the law $A+(B/\cos^2\frac{1}{2}\vartheta)$, it will be impossible to decide whether any violation of the theory exist at all, or, if it exists, is of the special kind (1), considered in the present paper. Indeed, in the latter case the quantities $A$ and $B$ may be determined from experiment, but no conclusions about the value of $\mu_e$ can be drawn from eq. (11), since these two equations, apart from $\mu_e$, contain two more unknown functions $F_N$ and $\Phi_N$. All that can be done if we know $A$ and $B$ is to obtain $F_N$ and $\Phi_N$ from (11), assuming something about the value $\mu_e$, for example, $\mu_e \equiv 0$, i.e., that there is no violation of the theory.

Thus, the non-punctiform character (1) of an electron considered in this paper cannot be detected in principle by studying the dependence on energy and angle of the elastic scattering cross-section of electrons on nucleons in the region of large $q$. The solution of this problem may be obtained in much more complex experiments: for example, by measuring a change in the polarization of electrons in scattering, or by studying the scattering of polarized electrons (or the polarization of scattered electrons) with polarized nucleons.

Some of the relevant formulas are given in the supplement.

It will be noted in conclusion that the most direct method of observing the non-punctiform character of an electron will be the scattering of electrons on electrons in crossing beams, planned by Panofsky *et al.* [9]). These experiments make it possible in principle to determine both formfactors $a_e(q^2)$ and $b_e(q^2)$. Yet technically they are very complicated.

As was shown above, it is by far simpler to observe effects connected with the second term in (1), i.e. with the anomalous magnetic moment of a non-punctiform electron; if electrodynamics is violated, and $\Gamma_\mu^{(e)} \neq \gamma_\mu$, this term in general should necessarily be present. Our purpose was to draw experimentators' attention to the fact these effects (if there are such at all in the region from $10^{-13}$ to $10^{-14}$ cm) may be rather simply observed in experimenting with elastic scattering of electrons on $\alpha$-particles

The authors are indebted to L. D. Landau for a number of essential remarks and to I. M. Shmushkevitch for discussions.

## Appendix

If prior to scattering the spins of the proton and the electron are directed along the unit vectors **N** and **n** respectively, and after scattering along **N'** and **n'**, the scattering cross-section for a non-punctiform electron has the form

$$\sigma_1 = \sigma_0[\alpha+\beta(\mathbf{N}\cdot\mathbf{N}')+\gamma(\mathbf{n}\cdot\mathbf{n}')+\delta(\mathbf{N}\cdot\mathbf{n})+\eta(\mathbf{N}\cdot\mathbf{n}')+\xi(\mathbf{N}\cdot\mathbf{n})+\kappa(\mathbf{N}'\cdot\mathbf{n}')]$$

where $\sigma_0$ has the value determined by eq. (5), and the coefficients $\alpha, \beta, \ldots, \kappa$ are the functions of $q^2$, $\varepsilon$ (or $\vartheta$); they may in principle be measured experimentally. The values of some of them are given below:

$$\gamma = \frac{2\varepsilon(\varepsilon-\Delta)}{2\varepsilon(\varepsilon-\Delta)-M\Delta}\left[\gamma_0+\frac{\gamma_1}{\varepsilon-\Delta}+\frac{\gamma_2}{\varepsilon(\varepsilon-\Delta)}+\frac{\gamma_3}{\varepsilon(\varepsilon-\Delta)^2}+\frac{\gamma_4}{\varepsilon^2(\varepsilon-\Delta)^2}\right],$$

$$\gamma_0 = X^{-1}\{4\alpha_N{}^2(1-\mu_e{}^2)+M\Delta X^2\beta_N{}^2(1-\mu_e{}^2)\},$$

$$\gamma_1 = 16M\sqrt{\frac{\Delta}{2M}}X^{-1}\mu_e\alpha_N{}^2,$$

$$\gamma_2 = -M\Delta X^{-1}\left\{6\alpha_N{}^2+8\sqrt{\frac{\Delta}{2M}}\mu_e\alpha_N{}^2-4\alpha_N{}^2\left(1-\frac{\Delta}{4M}\right)\mu_e{}^2+\Delta M\beta_N{}^2(1-\mu_e{}^2)\right\},$$

$$\gamma_3 = -8M^2\Delta\sqrt{\frac{\Delta}{2M}}X^{-1}\alpha_N{}^2\mu_e,$$

$$\gamma_4 = 2M^2\Delta^2X^{-1}\alpha_N{}^2\left(1+\sqrt{\frac{\Delta}{2M}}\mu_e\right)^2,$$

$$X = 1+\frac{\Delta}{2M}, \qquad \alpha_N = \left(a_N+\frac{\Delta}{2M}b_N\right)a_e(q^2), \qquad b_N = \frac{1}{2MX}(a_N+b_N)a_e(q^2),$$

$$\eta = \frac{2\varepsilon(\varepsilon-\Delta)}{2\varepsilon(\varepsilon-\Delta)-M\Delta}\left[\eta_0+\frac{\eta_1}{\varepsilon-\Delta}+\frac{\eta_2}{\varepsilon(\varepsilon-\Delta)}+\frac{\eta_3}{\varepsilon(\varepsilon-\Delta)^2}+\frac{\eta_4}{\varepsilon^2(\varepsilon-\Delta)^2}\right]\alpha_N\beta_N,$$

$$\eta_0 = -4M\sqrt{\frac{\Delta}{2M}}\mu_e,$$

$$\eta_1 = 2M\Delta(1-\mu_e^2),$$

$$\eta_2 = -M\Delta^2\left(1-\frac{6M}{\Delta}\sqrt{\frac{\Delta}{2M}}\mu_e-\mu_e^2\right),$$

$$\eta_3 = -M^2\Delta^2(1-\mu_e^2),$$

$$\eta_4 = -\tfrac{1}{2}M^2\Delta^3\left[1-\left(\sqrt{\frac{\Delta}{2M}}-\sqrt{\frac{2M}{\Delta}}\mu_e+\mu_e^2\right)\right].$$

*Note added in proof:* Chandell and Berlesson have recently obtained data showing that up to distances of $4\times10^{-14}$ cm (up to $q \approx 500$ MeV) the electron has no anomalous magnetic moment. The measurements were undertaken to determine whether the electron has an electric dipole moment. Chandell and Berlesson measured the cross section at angles 60° and 135° for $q$, in the range from 1.25 $f^{-1}$ to 2.25 $f^{-1}$. The dependence of the cross section of scattering of an electron with a dipole moment on $\alpha$-particles is determined by a formula quite similar to eq. (4) of our paper, if $\mu^2 = \lambda^2 q^2/m_e^2$ where $\lambda$ is a certain constant characterizing the magnitude of the dipole moment. It follows from the data thus obtained that for $q \leqq 0.25\ f^{-1}$, i.e., for distances exceeding $4\times10^{-14}$ cm, $\mu \leqq 0.1\pm0.1$. ($(1.0\pm1.0)\times10^{-4}$ is obtained for $\lambda$.) Thus, these experimental data taken together with the theoretical considerations set down above tend to show that the dimensions of an electron at any rate do not exceed $0.4\times10^{-13}$ cm.

The authors are indebted to Dr. Chandell for the communication of the results of the above mentioned measurements.

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10.B *Nuclear Physics* **13** (1959) 695—696; © *North-Holland Publishing Co., Amsterdam*

# NEWS ITEMS

## INTERNATIONAL CONFERENCE ON NUCLEAR FORCES AND THE FEW-NUCLEON PROBLEM

### (London, 8—11 July 1959)

This conference was held in the Physics Department, University College, London, on the 8—11 July 1959. It was initiated by Dr. L. Cranberg (Los Alamos) and Prof. H. S. W. Massey (University College) and, although the original intention was to discuss the properties of nuclei of mass 2—5, the eventual scope covered this topic together with a long session on the fundamental two-nucleon interaction, and also a session on the properties of nuclei with $A \geqq 5$ which included binding energy calculations using the Brueckner model.

The first of the five main sessions was concerned with the nuclear force problem and review papers were given by R. E. Marshak on the "phenomenological aspects of the two-nucleon interaction", Richard Wilson on the "experimental status of the two-nucleon interaction", K. A. Brueckner on "the meson theory of nuclear forces" and G. F. Chew on "the applications of dispersion relations to nucleon-nucleon scattering". Among the remaining papers there was a summary of the Japanese work by J. Iwadare and a report by G. Breit on extensive phase shift searches carried out at Yale. Discussion centred around the necessity and theoretical justification for the $\mathbf{L} \cdot \mathbf{S}$ force and on the use of the asymptotic meson potential for high $l$-waves within the context of both Schrödinger theory and dispersion relations.

"The scattering of nucleons by light nuclei at high energies" was the title of the second session and a review paper on the Impulse Approximation was given by H. McManus. This was followed by contributed papers on applications of the method to specialised reactions (L. Castillejo and R. Phillips) as well as discussion of the method itself (M. L. Goldberger and R. J. Glauber). Experimental papers on p—d and p—$\alpha$ scattering were presented from Harvard, Harwell and University College, London.

A short session on "photonuclear reactions" was followed by the session on "binding energies and elastic scattering of light nuclei at low energies". Two theoretical review papers from London were given by H. S. W. Massey and P. G. Burke on the 3—5 body problem. L. Cranberg and L. Rosen gave invited papers describing extensive experimental work at Los Alamos; these included several new checks of charge symmetry and time reversal invariance in low energy nuclear reactions. Among other papers presented were one by J. M. Blatt on the "ground state energy of the triton" (which is unbound for G. T. potentials and reasonable trial functions), one by J. L. Gammel on the "feasibility of spin correlation experiments using $He^4$ as analyzer" and one by G. Skornyakov on the exact integration of the n—d problem with zero range forces.

B. H. Bransden opened the last session with a review on the "collisions of neutrons and deuterons with $H^3$ and $He^3$". The application of the Resonating Group Structure method to binding energy and scattering calculations, as did the Impulse Approximation method in an earlier session, aroused considerable controversy. The other review paper in this session was by V. J. Emery on the Brueckner model applied to binding-energy calculations. Contributed papers included one by K. A. Brueckner on the "Hartree-Fock method for strongly interacting systems".

T. C. Griffith, E. A. Power

10.A

## THE INSTITUTE OF PHYSICS

At the Annual General Meeting of The Institute of Physics held in London on 7 July 1959, Sir George Thomson (Master of Corpus Christi College, Cambridge) was re-elected President.

Dr J. M. A. Lenihan was elected as Vice-President; Dr J. Taylor, Honorary Treasurer, and Professor F. A. Vick, Honorary Secretary, were re-elected. The two new Ordinary Members of the Board elected were Dr V. E. Cosslett and Mr L. Rotherham.

The Report for 1958 which was adopted at the meeting shows that at the end of the year the total membership in all grades was 6309, an increase of 415 on the previous year. 851 applications for election or transfer to the various grades were considered, which was 7 % more than the previous year.

The Report records that at the end of the year a circular was sent to members proposing the amalgamation of the Institute and The Physical Society and that if the broad basis of intention was accepted a Joint Committee would be set up to work out all the details; it is understood that a Scheme for amalgamation is about to be submitted to members of both bodies.

Eighty candidates took the recently established Graduateship Examination, and twenty-six satisfied the examiners. In 1958, 637 students took the Ordinary National Certificate examination and 246 the Higher National Certificate examination in applied physics (compared with 542 and 182 respectively in 1957).

Besides the normal work of the Institute the Report records expanding activities in many fields including education, scientific publications, conferences and symposia and other meetings arranged by the Institute's Branches at home and oversea and by its specialist subject Groups.

The Institute's Benevolent Fund continued to help those of its members and dependents who had met with misfortune.

The address of the Institute of Physics is 47 Belgrave Square, London, S.W. 1.

10.E *Nuclear Physics* **13** (1959) 697; © *North-Holland Publishing Co., Amsterdam*

# ERRATUM

G. R. Satchler, Deuteron stripping with Polarised Deuterons, Nuclear Physics **6** (1958) 543.

The last factor of equation (1) should read

$$(1+3\mathbf{P}_p \cdot \mathbf{P}_d).$$

The author is indebted to Dr. B. Hird for remarking that the physical reason for the positive sign is clear. The proton spin within the incident deuteron tends to be parallel to the total deuteron spin. Thus, if the release of protons with their spins along $\mathbf{P}_p$ is favoured, the cross-section will be enhanced if the incident deuteron spin is itself parallel to $\mathbf{P}_p$.

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